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On the Chapman-Kolmogorov Equation

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ON THE CHAPMAN–KOLMOGOROV EQUATION

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CONTENTS

In the form considered in this paper, the Chapman–Kolmogorov equation connects a doubly infinite collection of functions, and imposes complex constraints on each of them. The main theorems characterize the functions which satisfy these constraints, and generalize known results in the theory of continuous-time Markov chains.

1. Semigroups of positive matrices

1.1. *Introduction*

The starting-point of this paper is the equation

$$p_{ij}(s+t) = \sum_{k\in I} p_{ik}(s)\, p_{kj}(t) \quad (i,j\in I;\ s,t>0), \tag{1.1.1}$$

where I is a countable set and, for each i, j in I, p_{ij} is a positive† Lebesgue measurable function on the half-line $(0,\infty)$. This is often called the Chapman–Kolmogorov equation, after the late S. Chapman, F.R.S., who used a continuous version of it in (Chapman 1928), and A. N. Kolmogorov, For.Mem.R.S., in whose hands it first revealed its depth and subtlety.

Collections of functions satisfying (1.1.1) are central to the theory of continuous-time Markov chains, in which context they have been studied for many years. In that theory (of which the best account is by Chung (1967)) it is usual to impose the additional condition

$$\sum_{j\in I} p_{ij}(t) \leqslant 1 \quad (i\in I;\ t>0), \tag{1.1.2}$$

and most of the methods used depend in an essential way on this condition.

† Throughout this paper the words positive, negative, increasing and decreasing are to be understood in the weak sense unless preceded by the adverb strictly. Thus x is positive if $x \geqslant 0$, and strictly positive if $x > 0$.

It was, however, pointed out by Jurkat (1959, 1960) that many of the results of the Markov theory could, with ingenuity, be proved without recourse to (1.1.2). He assumed the continuity condition

$$\lim_{t\to 0} p_{ij}(t) = \delta_{ij} \tag{1.1.3}$$

(where $\delta_{ij} = 1$ or 0 according as $i = j$ or $i \neq j$) and showed, for instance, that the functions p_{ij} are continuously differentiable in $(0, \infty)$. His programme was continued by Chung (1963), who (in theorem 10.1) established in full generality the *Lévy dichotomy*: each function p_{ij} is either strictly positive or identically zero on $(0, \infty)$.

Recent work of Cornish (1971) has shown that Jurkat's continuity condition (1.1.3) represents no real loss of generality. He shows that, unless there exists $a \in I$ with

$$p_{ia} = 0 \quad \text{for all } i \quad \textit{or} \quad p_{aj} = 0 \quad \text{for all } j, \tag{1.1.4}$$

then there are positive numbers α_i, β_i $(i \in I)$, a function f from I on to a set $\tilde{I}$, and a collection of functions $(\tilde{p}_{ij};\, i, j \in \tilde{I})$ satisfying (1.1.1) and (1.1.3), such that

$$p_{ij}(t) = \alpha_i \beta_j \tilde{p}_{f(i)f(j)}(t). \tag{1.1.5}$$

Hence the study of (p_{ij}) reduces to that of the system $(\tilde{p}_{ij})$, for which the continuity condition (1.1.3) is available. Moreover, if there are anomalous elements satisfying (1.1.4), these may be removed from I without affecting (1.1.1) (and then correspond to 'entrance' or 'exit' laws (Neveu 1961) for the reduced collection). Hence in the present analysis (1.1.3) will be assumed without further comment.

The Chapman–Kolmogorov equation asserts, in effect, that if the functions p_{ij} are arranged in a matrix

$$\boldsymbol{P}(t) = (p_{ij}(t);\, i, j \in I), \tag{1.1.6}$$

then

$$\boldsymbol{P}(s+t) = \boldsymbol{P}(s)\, \boldsymbol{P}(t) \quad (s, t > 0), \tag{1.1.7}$$

so that the $\boldsymbol{P}(t)$ form a one-parameter semigroup of (finite or infinite) positive matrices. If such a semigroup satisfies (1.1.1), (1.1.2) and (1.1.3) it is usually called a *Markov semigroup*; if it is only known to satisfy (1.1.1) and (1.1.3) it will here be described as an *M-semigroup*. Thus Jurkat's contribution was to show that a number of the results known to be true of Markov semigroups hold also for the more extensive class of M-semigroups.

This observation is important from the purely mathematical point of view, not least because the form of (1.1.3), despite its natural probabilistic meaning, is rather artificial. It means, for example, that the transpose of a Markov semigroup (obtained by interchanging i and j) is not, in general, a Markov semigroup, and the theory is thereby robbed of a natural duality. But there are also practical reasons for eschewing (1.1.2). For example, the theory of continuous-time multitype branching processes (Harris 1963, § V. 15) gives rise to expectation matrices forming M-semigroups which are not usually Markov semigroups.

On the other hand, many M-semigroups can be turned into Markov semigroups by a simple transformation. It was shown in Kingman (1963) that, if an M-semigroup is irreducible in the sense that none of its functions p_{ij} vanishes identically, and if there is a constant β such that, for some i,

$$p_{ii}(t) = O(e^{\beta t}) \tag{1.1.8}$$

as $t \to \infty$, then there are positive numbers x_i $(i \in I)$ such that the functions

$$\bar{p}_{ij}(t) = e^{-\beta t} p_{ij}(t)\, x_j / x_i \tag{1.1.9}$$

form a Markov semigroup. We say that such semigroups are *tame*, and it is clear that M-semigroups are only really different from Markov semigroups when they are *wild* in the sense that no transformation of the form (1.1.9) makes them satisfy (1.1.2).

An important technique in the theory of Markov semigroups is the Hille–Phillips theory of one-parameter semigroups of operators (Hille & Phillips 1957; Kendall & Reuter 1954). This can be applied to M-semigroups if, but apparently only if, they are tame in the sense just defined. A theory of infinitesimal generators of wild M-semigroups is not at present available, but would be of great interest.

When I is finite, a simple argument shows that all M-semigroups are tame. When I is countably infinite, on the other hand, the existence of wild M-semigroups was established by Cornish (1971), who produced examples to show that $p_{ii}(t)$ could be made to increase, as $t \to \infty$, faster than any given function of t. Further examples may be found in Kingman (1973).

The object of this paper is to examine M-semigroups, and more especially the functions p_{ij} of which they consist, from the point of view used in Kingman (1972) to analyse the Markov case. Jurkat's results will not be assumed, since they emerge naturally as by-products of the present development. Indeed, the only result required from the existing literature on M-semigroups (apart from the Cornish apologia for (1.1.3) already quoted) will be the Lévy dichotomy, in the form proved by Chung (1963).

1.2. *Semi-p-functions*

The classical theory of Markov semigroups contains many results which bear upon the properties of the individual functions p_{ij}, and these taken together strongly suggest the problem of characterizing such functions. This problem has two aspects, since the behaviour of p_{ij} is quite different when $i = j$ from that when $i \neq j$ (witness (1.1.3)). Both were resolved in Kingman (1971), although the result is more satisfactory in the diagonal case $i = j$, for which it gives an effective algorithm, than in the non-diagonal case $i \neq j$. To what extent can these results be generalized to M-semigroups?

Confining attention for the moment to the diagonal problem, we therefore have the following general question:

Given a function p: $(0, \infty) \to [0, \infty)$, under what conditions on p does there exist an M-semigroup and an index i such that $p_{ii}(t) = p(t)$ for all $t > 0$?

This question will be answered by generalizing the techniques used to resolve the corresponding problem for Markov semigroups (of which there is a connected account in Kingman (1972)). For brevity, a function expressible in the form p_{ii} in some M-semigroup will be said to enjoy *property M*.

The first step in characterizing functions with property M is to notice that, in any M-semigroup, the function p_{ii} (for fixed $i \in I$) satisfies certain inequalities which flow from the Chapman–Kolmogorov equation, together with the positive character of (the elements of) the matrix $\boldsymbol{P}(t)$. The simplest of these comes from setting $j = i$ in (1.1.1), so that

$$p_{ii}(s+t) = \sum_{k \in I} p_{ik}(s)\, p_{ki}(t) \geqslant p_{ii}(s)\, p_{ii}(t).$$

Hence a necessary condition for a function p to have property M is that

$$p(s+t) \geqslant p(s)\, p(t) \quad (s, t > 0). \tag{1.2.1}$$

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Another necessary condition, this time from (1.1.3), is that

$$\lim_{t\to 0} p(t) = 1, \tag{1.2.2}$$

and it should be noted that these together imply that

$$p(t) > 0 \quad (t > 0). \tag{1.2.3}$$

The inequality (1.2.1) is the first of an infinite family which may be generated as follows. Let

$$0 = t_0 < t_1 < t_2 < \dots \tag{1.2.4}$$

and write (for $n \geqslant 1$)

$$f_n = \Sigma p_{ik_1}(t_1)\, p_{k_1 k_2}(t_2 - t_1) \dots p_{k_{n-1} i}(t_n - t_{n-1}), \tag{1.2.5}$$

where the summation extends over all $k_1, k_2, \dots, k_{n-1}$ not equal to i. Had the summation extended over all $k_1, k_2, \dots, k_{n-1}$, the result would (by $n-1$ applications of (1.1.1)) have been $p_{ii}(t_n)$. Now this latter sum could be split up into n subsums $\Sigma_1, \Sigma_2, \dots, \Sigma_n$, where Σ_r extends over all $k_1, k_2, \dots, k_{n-1}$ with $k_1, k_2, \dots, k_{r-1} \neq i$ and $k_r = i$, and clearly

$$\begin{aligned}\Sigma_r &= f_r \Sigma p_{ik_{r+1}}(t_{r+1} - t_r)\, p_{k_{r+1} k_{r+2}}(t_{r+2} - t_{r+1}) \dots p_{k_{n-1} i}(t_n - t_{n-1}) \\ &= f_r p_{ii}(t_n - t_r).\end{aligned}$$

Hence, for all $n \geqslant 1$,

$$p_{ii}(t_n) = \sum_{r=1}^{n} f_r p_{ii}(t_n - t_r). \tag{1.2.6}$$

If the sequence (1.2.4) is fixed, (1.2.6) may be solved for f_n in terms of the values of p_{ii}; by Cramer's rule

$$f_n = F(t_1, t_2, \dots, t_n; p_{ii}), \tag{1.2.7}$$

where F is defined by

$$F(t_1, t_2, \dots, t_n; p) = (-1)^{n-1} \begin{vmatrix} p(t_1) & 1 & 0 & 0 & \dots & 0 \\ p(t_2) & p(t_2 - t_1) & 1 & 0 & \dots & 0 \\ p(t_3) & p(t_3 - t_1) & p(t_3 - t_2) & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p(t_n) & p(t_n - t_1) & p(t_n - t_2) & p(t_n - t_3) & \dots & p(t_n - t_{n-1}) \end{vmatrix}. \tag{1.2.8}$$

It is evident from (1.2.5) that $f_n \geqslant 0$, so that a necessary condition for a function p to have property M is that

$$F(t_1, t_2, \dots, t_n; p) \geqslant 0 \quad (n = 1, 2, 3, \dots) \tag{1.2.9}$$

for any sequence of the form (1.2.4). The case $n = 1$ is trivial, while $n = 2$ is a restatement of (1.2.1).

Functions satisfying (1.2.9) together with the inequality

$$\sum_{n=1}^{\infty} F(t_1, t_2, \dots, t_n; p) \leqslant 1 \tag{1.2.10}$$

have been studied under the name of p-functions, and their theory is described in detail by Kingman (1972). In the present context (1.2.10), which is a consequence of (1.1.2), is not available, and we must be content with (1.2.9) alone. A function satisfying (1.2.9) is called a *semi-p-function*, and such functions have been considered by Kingman (1973). The theory of semi-p-functions is inevitably more difficult than that of p-functions, for the same reason that it is difficult to cut a sheet of paper with one blade of a pair of scissors. The fundamental result is that every semi-p-function satisfying (1.2.2) is the unique solution of the Volterra equation,

$$p(t) = 1-\int_0^t p(t-s)\,m(s)\,\mathrm{d}s, \tag{1.2.11}$$

for some decreasing function m on $(0, \infty)$ which is integrable on $(0, 1)$. The function m is uniquely determined by p except at its discontinuities, and every such function m determines a semi-p-function.

A semi-p-function p is a p-function (that is, it satisfies (1.2.10) as well as (1.2.9)) if and only if $p(t) \leqslant 1$ for all $t > 0$, and this in turn is true if and only if $m(t) \geqslant 0$ for all $t > 0$. Kingman (1973) proves that every semi-p-function satisfying (1.2.2) coincides on every finite interval with the product of a p-function and an exponential function. Hence the local properties of semi-p-functions can at once be deduced from those of p-functions: for example, the semi-p-function p is continuous and has finite left and right derivatives in $(0, \infty)$, which differ only on the countable set of discontinuities of m, and the limit

$$\lim_{t\to 0} t^{-1}\{1-p(t)\} = \lim_{t\to 0} m(t) \leqslant \infty \tag{1.2.12}$$

exists.

In what follows, the only semi-p-functions considered will be those satisfying (1.2.2), and this condition will be assumed without further comment, as will the convention

$$p(0) = 1. \tag{1.2.13}$$

1.3. *The main results*

A function having property M must be a semi-p-function, but (as examples from the Markov case show) there are semi-p-functions which do not enjoy this property. Thus the characterization problem remains, but the theory of semi-p-functions allows it to be thrown into a different form. The Volterra equation (1.2.11) sets up a one-to-one correspondence between the class of semi-p-functions p and the class of right-continuous, decreasing, locally integrable functions m. It will be shown in theorem VII that, in this correspondence, those semi-p-functions having property M correspond exactly to those functions m for which there exists a lower semicontinuous function h with

$$m(s)-m(t) = \int_s^t h(x)\,\mathrm{d}x, \tag{1.3.1}$$

where h is either identically zero or is strictly positive and satisfies

$$h(x) \geqslant \mathrm{e}^{-\beta x} \tag{1.3.2}$$

for some constant β and all sufficiently large x.

This result is a complete solution of the problem posed at the beginning of § 1.2, although the intervention of the Volterra equation can raise difficulties in applying the result, as indeed it does in the Markov case (cf. Kingman 1972, ch. 6). It has one very important corollary: if a function p has property M, and if $T > 0$, then there is a *tame* M-semigroup with

$$p_{ii}(t) = p(t) \quad (0 \leqslant t \leqslant T)$$

for some i. Since tame semigroups are virtually within the scope of the classical Markov theory, this means that local properties of the diagonal elements of M-semigroups follow at once from the corresponding results for Markov semigroups.

This 'localization principle' has a far-reaching generalization which, among other things, enables the local properties of the non-diagonal elements of M-semigroups to be deduced from the Markov theory. Theorem X asserts that, if $\boldsymbol{P}(t) = (p_{ij}(t); i,j \in I)$ is an M-semigroup, $T > 0$, and J is a finite subset of I, then there exists a Markov semigroup $\tilde{\boldsymbol{P}}(t) = (\tilde{p}_{ij}(t); i,j \in I)$ and a constant α such that

$$p_{ij}(t) = \tilde{p}_{ij}(t)\, e^{\alpha t} \quad (i,j \in J;\ 0 \leqslant t \leqslant T). \tag{1.3.3}$$

Thus, in order to discover whether an M-semigroup is wild, it is necessary to know about $p_{ij}(t)$ either for infinitely many pairs (i,j), or for arbitrarily large values of t. No finite submatrix of $\boldsymbol{P}(t)$, observed on a finite interval of t-values, contains the seeds of wildness.

In order to establish these results, the essential tool is the concept of a *semi-p-matrix*. This generalizes both the idea of a p-matrix (Kingman 1965) and that of a semi-p-function; the diagram

$$\begin{array}{ccc} \text{p-function} & \longrightarrow & \text{semi-p-function} \\ \downarrow & & \downarrow \\ \text{p-matrix} & \longrightarrow & \text{semi-p-matrix} \end{array}$$

in which the arrows denote increasing generality, summarizes the situation. A structure theorem for semi-p-matrices (theorem II) is proved, and used to establish the decompositions necessary to prove the deeper properties of functions with property M. That these properties are necessary is the content of theorem VI; that they are also sufficient is asserted in theorem VII.

The most important example of a semi-p-matrix is a finite principal submatrix of an M-semigroup, and theorem VIII generalizes theorem VII by characterizing those semi-p-matrices which can arise in this way. The localization principle follows, with its consequences for the non-diagonal elements of M-semigroups.

In Kingman (1965) I suggested a classification describing the relative depth of results about the elements of Markov semigroups, and this too extends in a natural way to M-semigroups:

Type I. Results which follow from the fact that p_{ii} is a semi-p-function (such as the existence of right and left derivatives).

Type II. Results which follow from the fact that, for $i \neq j$, the matrix

$$\begin{pmatrix} p_{ii}(t) & p_{ij}(t) \\ p_{ji}(t) & p_{jj}(t) \end{pmatrix} \tag{1.3.4}$$

in a semi-p-matrix – such as the first passage and last exit decompositions (§ 2.4) and the existence of the finite limit

$$q_{ij} = \lim_{t \to 0} p_{ij}(t)/t. \tag{1.3.5}$$

Type III. Results which cannot be deduced from the theory of semi-p-matrices alone.

It is, of course, the existence of the deep type III results which makes the characterization problem interesting and difficult. The most striking is Chung's version of the Lévy dichotomy, and another is Jurkat's generalization of Ornstein's theorem, which asserts the continuous differentiability of p_{ij} on $(0, \infty)$.

The culmination of this paper is the characterization theorem (theorem VIII) for principal $(N \times N)$ submatrices of M-semigroups. Had we been content with the special case $N = 1$, for which theorem VII is adequate, some of the deeper aspects of the theory of semi-p-matrices could have been omitted. Specifically, only (2×2) semi-p-matrices need have been considered, with a corresponding simplification of the decomposition theory of § 2.4, and the theory of taboo semigroups (§ 3.1) would not have been required. But as soon as properties of non-diagonal elements are sought, the more general theory is essential.

Finally, a technical remark; the reader familiar with the Markov theory will recall its heavy dependence on Laplace transform techniques. These are not available in the present context, since an element of a wild M-semigroup typically has a nowhere convergent Laplace transform. This inconvenient fact means that transform manipulations must sometimes be replaced by less transparent, if more direct, calculations. Their structure may sometimes be more readily appreciated by an illicit formal application of the Laplace transform.

2. The theory of semi-p-matrices

2.1. *Definition*

Let $\boldsymbol{P}(t) = (p_{ij}(t); i,j \in I)$ be an M-semigroup with index set I, and let $J = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of I. For each $t > 0$, let $\boldsymbol{p}(t)$ denote the corresponding principal submatrix of $\boldsymbol{P}(t)$:

$$\boldsymbol{p}(t) = (p_{x_i x_j}(t); i,j = 1, 2, \ldots, N). \tag{2.1.1}$$

Then the $(N \times N)$ matrix-valued function $\boldsymbol{p}$ satisfies an infinite family of inequalities which reduce to (1.2.9) when $N = 1$.

For any sequence (1.2.4) and any i, j in $1 \leqslant i, j \leqslant N$, write

$$f_{nij} = \Sigma\, p_{x_i y_1}(t_1)\, p_{y_1 y_2}(t_2 - t_1) \ldots p_{y_{n-2} y_{n-1}}(t_{n-1} - t_{n-2})\, p_{y_{n-1} x_j}(t_n - t_{n-1}), \tag{2.1.2}$$

where the summation extends over $y_1, y_2, \ldots, y_{n-1} \notin J$. Then the argument leading to (1.2.6) generalizes in an obvious way to give, for $n \geqslant 1$,

$$p_{x_i x_j}(t_n) = \sum_{k=1}^{N} \sum_{r=1}^{n} f_{rik} p_{y_k x_j}(t_n - t_r), \tag{2.1.3}$$

or in matrix notation

$$\boldsymbol{p}(t_n) = \sum_{r=1}^{n} \boldsymbol{f}_r \boldsymbol{p}(t_n - t_r), \tag{2.1.4}$$

with the convention that

$$\boldsymbol{p}(0) = \boldsymbol{I}, \tag{2.1.5}$$

the identity matrix of order N. Writing (2.1.4) in the form

$$\boldsymbol{f}_n = \boldsymbol{p}(t_n) - \sum_{r=1}^{n-1} \boldsymbol{f}_r \boldsymbol{p}(t_n - t_r) \quad (n \geqslant 1),$$

it determines $\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3, \ldots$ recursively in terms of the values of $\boldsymbol{p}$:

$$\boldsymbol{f}_n = \boldsymbol{F}(t_1, t_2, \ldots, t_n; \boldsymbol{p}). \tag{2.1.6}$$

The precise form of $\boldsymbol{F}$ is not of importance for the present argument, but it is in fact easy to verify that

$$\boldsymbol{F}(t_1, t_2, \ldots, t_n; \boldsymbol{p}) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{0=\alpha_0<\alpha_1<\ldots<\alpha_k=n} \prod_{l=1}^{k} \boldsymbol{p}(t_{\alpha_l} - t_{\alpha_{l-1}}). \tag{2.1.7}$$

Alternatively, the determinantal expression (1.2.8) continues to hold, with p replaced by $\boldsymbol{p}$, F by $\boldsymbol{F}$ and 1 by $\boldsymbol{I}$, so long as the products implicit in the determinant are read in the order of the columns.

From (2.1.2), $f_{nij} \geqslant 0$, and it follows that the matrix-valued function $\boldsymbol{p}$ defined by (2.1.1) satisfies the inequalities

$$\boldsymbol{F}(t_1, t_2, \ldots, t_n; \boldsymbol{p}) \geqslant \boldsymbol{0} \quad (n = 1, 2, 3, \ldots), \tag{2.1.8}$$

where inequalities between matrices (and likewise between vectors) are interpreted, as they will be throughout this paper, element by element. In the Markov theory (Kingman 1965) there is a reverse inequality generalizing (1.2.10):

$$\sum_{n=1}^{\infty} F(t_1, t_2, \ldots, t_n; \boldsymbol{p})\,\mathbf{1} \leqslant \mathbf{1}, \tag{2.1.9}$$

where $\mathbf{1}$ is the vector all of whose components are equal to 1. A funtion $\boldsymbol{p}$ from $(0, \infty)$ into the space of $(N \times N)$ real matrices which satisfies (2.1.8) and (2.1.9) for all sequences (1.2.4) is called a *p-matrix*. If only (2.1.8) is known to hold, p is called a *semi-p-matrix*. Thus a finite principal submatrix of a Markov semigroup is a p-matrix, and of an M-semigroup is a semi-p-matrix.

In this paper, all semi-p-matrices will satisfy (as does (2.2.1) because of (1.1.3)) the continuity condition

$$\lim_{t\to 0} \boldsymbol{p}(t) = \boldsymbol{I} \tag{2.1.10}$$

and this, with the convention (2.1.5), will be assumed without further comment.

2.2. *The taming theorem*

In the theory of semi-p-functions (Kingman 1973) the key result is one which states that, on every finite interval, a semi-p-function is indistinguishable from the product of an exponential function and a p-function. This is also true (though rather more difficult to prove) for semi-p-matrices.

Theorem I. *Let* $\boldsymbol{p}$ *be a function from* $(0, \infty)$ *into the space of* $(N \times N)$ *matrices, which satisfies* (2.1.8) *and* (2.1.10). *Then, for any finite positive* T, *there exists a finite number* α *and a p-matrix* $\tilde{\boldsymbol{p}}$ *such that*

$$\boldsymbol{p}(t) = \tilde{\boldsymbol{p}}(t)\,e^{\alpha t} \quad (0 < t \leqslant T). \tag{2.2.1}$$

Proof. Setting $n = 2$ in (2.1.8) shows that

$$p_{ij}(t_2) \geqslant \sum_{k=1}^{N} p_{ik}(t_1)\,p_{kj}(t_2 - t_1) \geqslant p_{ij}(t_1)\,p_{jj}(t_2 - t_1) \tag{2.2.2}$$

for $0 < t_1 < t_2$. Since $p_{jj}(t) \to 1$ as $t \to 0$, it follows on letting $t_2 - t_1 \to 0$, first with t_1 fixed and then with t_2 fixed, that p_{ij} has right and left limits $p_{ij}(t+)$ and $p_{ij}(t-)$ at each $t > 0$, and that

$$p_{ij}(t-) \leqslant p_{ij}(t) \leqslant p_{ij}(t+). \tag{2.2.3}$$

(See Jurkat (1959) for the details of this argument.) Now

$$0 \leqslant F_{ij}(t_1, t_2, t_3; \boldsymbol{p})$$
$$= p_{ij}(t_3) - \sum_{k=1}^{N} p_{ik}(t_1)\,p_{kj}(t_3 - t_1) - \sum_{k=1}^{N} p_{ik}(t_2)\,p_{kj}(t_3 - t_2) + \sum_{k,l=1}^{N} p_{ik}(t_1)\,p_{kl}(t_2 - t_1)\,p_{lj}(t_3 - t_2).$$

Letting $t_3 \to t_2$ we have

$$p_{ij}(t_2+) - p_{ij}(t_2) \geqslant \sum_{k=1}^{N} p_{ik}(t_1)\,\{p_{kj}(t_2 - t_1 +) - p_{kj}(t_2 - t_1)\}$$
$$\geqslant p_{ii}(t_1)\,\{p_{ij}(t_2 - t_1 +) - p_{ij}(t_2 - t_1)\}. \tag{2.2.4}$$

For any $t > 0$, since p_{ij} has only jump discontinuities, it is continuous off a countable set, and hence we can choose $t_2 > t$ so that $p_{ij}(t_2+) = p_{ij}(t_2)$. Moreover, (2.2.2) with $i = j$, combined with (2.1.10), shows that $p_{ii} > 0$. Hence (2.2.4) with $t_1 = t_2 - t$ implies that $p_{ij}(t+) = p_{ij}(t)$.

An exactly similar argument, starting by letting $t_1 \to t_2$, shows that $p_{ij}(t) = p_{ij}(t-)$. Hence p_{ij} is continuous in $(0, \infty)$, and indeed in $[0, \infty)$ because of (2.1.10) and (2.1.5).

Now consider the functions

$$\phi_i(t) = \int_t^{t+1} \sum_{j=1}^{N} p_{ij}(s)\, \mathrm{d}s \quad (1 \leqslant i \leqslant N,\ t \geqslant 0),$$

which are strictly positive (since p_{ii} is) and continuously differentiable. Then the functions

$$t \mapsto \frac{\log \phi_i(T) - \log \phi_i(t)}{T-t}$$

are bounded on $[0, T)$ and hence we may choose a positive number β (depending of course on T) so large that

$$\phi_i(t)\, \mathrm{e}^{-\beta t} \geqslant \phi_i(T)\, \mathrm{e}^{-\beta T} \tag{2.2.5}$$

for all $0 \leqslant t < T$ and all $1 \leqslant i \leqslant N$. If

$$0 = t_0 < t_1 < t_2 < \ldots < t_n < T < s,$$

then (2.1.4) shows that

$$p_{ij}(s) = \sum_{r=1}^{n} \sum_{k=1}^{N} F_{ik}(t_1, \ldots, t_r; \boldsymbol{p})\, p_{kj}(s-t_r) + F_{ij}(t_1, \ldots, t_n, s; \boldsymbol{p}),$$

so that

$$\sum_{j=1}^{N} p_{ij}(s) \geqslant \sum_{r=1}^{n} \sum_{k,j=1}^{N} F_{ik}(t_1, \ldots, t_r; \boldsymbol{p})\, p_{kj}(s-t_r).$$

Integrating from $s = T$ to $s = T+1$, we have

$$\begin{aligned}\phi_i(T) &\geqslant \sum_{r=1}^{n} \sum_{k=1}^{N} F_{ik}(t_1, \ldots, t_r; \boldsymbol{p})\, \phi_k(T-t_r) \\ &\geqslant \sum_{r=1}^{n} \sum_{k=1}^{N} F_{ik}(t_1, \ldots, t_r; \boldsymbol{p})\, \phi_k(T)\, \mathrm{e}^{-\beta t_r},\end{aligned}$$

using (2.2.5). Hence, if we define

$$\bar{p}_{ij}(t) = p_{ij}(t)\, \phi_j(T)\, \mathrm{e}^{-\beta t}/\phi_i(T) \quad (0 \leqslant t \leqslant T), \tag{2.2.6}$$

and note that (2.1.7) implies that

$$F_{ij}(t_1, \ldots, t_r; \bar{\boldsymbol{p}}) = F_{ij}(t_1, \ldots, t_r; \boldsymbol{p})\, \phi_j(T)\, \mathrm{e}^{-\beta t_r}/\phi_i(T), \tag{2.2.7}$$

then

$$\sum_{r=1}^{n} F(t_1, \ldots, t_r; \bar{\boldsymbol{p}})\, \mathbf{1} \leqslant \mathbf{1} \tag{2.2.8}$$

for any sequence (1.2.4) with $t_n \leqslant T$.

Note that $\bar{\boldsymbol{p}}$ has so far only been defined on $[0. T]$, and our next task is to extend its definition to $[0, \infty)$ in such a way that (2.2.8) holds even if $t_n > T$. To do this, let n be a positive integer, and write

$$f_{ij}(r) = F_{ij}(T/n, 2T/n, \ldots, rT/n; \bar{\boldsymbol{p}}),$$

$$g_i(r) = 1 - \sum_{s=1}^{r} \sum_{j=1}^{N} f_{ij}(s), \quad g_i(0) = 1,$$

for $1 \leqslant r \leqslant n$, $1 \leqslant i, j \leqslant N$, noting that (2.2.7) and (2.2.8) imply that

$$f_{ij}(r) \geqslant 0, \quad g_i(r) \geqslant 0.$$

Let $I = \{1, 2, \ldots, N\} \times \{1, 2, \ldots, n\}$, and define a stochastic matrix

$$\boldsymbol{\Pi} = (\pi_{(i,r),(j,s)};\ 1 \leqslant i, j \leqslant N,\ 1 \leqslant r, s \leqslant n)$$

on I by

$$\pi_{(i,r),(i,r+1)} = g_i(r)/g_i(r-1) \quad (1 \leqslant i \leqslant N,\ 1 \leqslant r \leqslant n-1),$$

$$\pi_{(i,r),(j,1)} = f_{ij}(r)/g_i(r-1) \quad (1 \leqslant i, j \leqslant N,\ 1 \leqslant r \leqslant n-1),$$

$$\pi_{(i,n),(i,n)} = 1 \quad (1 \leqslant i \leqslant N),$$

all other elements of $\boldsymbol{\Pi}$ being zero. (If it should happen that $g_i(s) = 0$ for some s, then indeterminate expressions $0/0$ may be given any values which leave $\boldsymbol{\Pi}$ stochastic.) Then an easy induction on k shows that

$$(\boldsymbol{\Pi}^k)_{(i,1),(j,1)} = \bar{p}_{ij}(kT/N). \tag{2.2.9}$$

It is a familiar fact, and easily verified, that for any stochastic matrix $\boldsymbol{\Pi}$, and any positive constant κ, the expression

$$\boldsymbol{P}(t) = \sum_{k=0}^{\infty} \frac{(\kappa t)^k\, \mathrm{e}^{-\kappa t}}{k!} \boldsymbol{\Pi}^k$$

defines a Markov semigroup. Let $\kappa = n/T$, and consider the p-matrix, which since it depends on n will be written $\boldsymbol{p}_n$, defined by (2.1.1) with $x_i = (i, 1)$. Then (2.2.9) shows that

$$(\boldsymbol{p}_n(t))_{ij} = \sum_{k=0}^{\infty} \frac{(\kappa t)^k\, \mathrm{e}^{-\kappa t}}{k!} (\boldsymbol{\Pi}^k)_{(i,1),(j,1)}$$

$$= \sum_{k=0}^{n} \frac{(nt/T)^k\, \mathrm{e}^{-nt/T}}{k!} \bar{p}_{ij}(kT/n) + \Delta,$$

where

$$0 \leqslant \Delta \leqslant \sum_{k=n+1}^{\infty} (nt/T)^k\, \mathrm{e}^{-kt/T}/k!$$

A familiar approximation argument (cf. Kingman 1972, p. 36), using the continuity of $\bar{\boldsymbol{p}}$ on $[0, T]$, shows that

$$\lim_{n\to\infty} \boldsymbol{p}_n(t) = \bar{\boldsymbol{p}}(t) \quad (0 \leqslant t < T), \tag{2.2.10}$$

where limits of matrices are taken element by element.

Hence we have constructed a sequence of p-matrices $\boldsymbol{p}_n$ whose values converge, for each $t < T$, to those of $\bar{\boldsymbol{p}}$. Regard $\boldsymbol{p}_n$ as an element of the space Ω of functions from $[0, \infty)$ into the compact space of $(N \times N)$ matrices with elements in $[0, 1]$. In its product topology Ω is compact (Tychonov's theorem), and hence the sequence $(\boldsymbol{p}_n)$ has a limit point $\boldsymbol{p}^*$, and (2.2.10) shows that

$$\boldsymbol{p}^*(t) = \bar{\boldsymbol{p}}(t) \quad (0 \leqslant t < T). \tag{2.2.11}$$

The functions

$$\boldsymbol{p} \mapsto \boldsymbol{F}(t_1, t_2, \ldots, t_n; \boldsymbol{p})$$

are continuous with respect to the product topology on Ω, and since $\boldsymbol{p}_n$ satisfies (2.1.8) and (2.1.9) so does $\boldsymbol{p}^*$. Moreover, (2.2.11) shows that $\boldsymbol{p}^*$ satisfies (2.1.10). Hence $\boldsymbol{p}^*$ is a p-matrix extending $\bar{\boldsymbol{p}}$.

Applying the structure theorem for p-matrices (Kingman 1972, theorem 5.2), there are measures μ_i^* on $(0, \infty]$ and λ_{ij}^* on $[0, \infty)$ such that, for all $\theta > 0$,

$$\int_0^{\infty} \boldsymbol{p}^*(t)\, \mathrm{e}^{-\theta t}\, \mathrm{d}t = \boldsymbol{q}^*(\theta)^{-1},$$

where $$q^*_{ii}(\theta) = \theta + \int_{(0,\infty]} (1 - e^{-\theta x})\, \mu^*_i(dx),$$

$$q^*_{ij}(\theta) = -\int_{[0,\infty)} e^{-\theta x}\, \lambda^*_{ij}(dx) \quad (i \neq j). \tag{2.2.12}$$

Define $$\tilde{p}_{ij}(t) = p^*_{ij}(t)\, \phi_i(T)\, e^{-\gamma t}/\phi_j(T), \tag{2.2.13}$$

where the positive constant γ is yet to be determined. Then

$$\int_0^\infty \tilde{\boldsymbol{p}}(t)\, e^{-\theta t}\, dt = \boldsymbol{q}(\theta)^{-1},$$

where $$q_{ij}(\theta) = q^*_{ij}(\theta + \gamma)\, \phi_i(T)/\phi_j(T) \quad (\theta > 0).$$

The structure theorem then shows easily that $\tilde{\boldsymbol{p}}$ is a p-matrix if and only if

$$\lim_{\theta \to 0} \sum_{j=1}^{N} q_{ij}(\theta) \geqslant 0$$

for all i (this being just another way of writing (5.2.5) of Kingman (1972)), or equivalently if and only if

$$\sum_{j=1}^{N} q^*_{ij}(\gamma)\, \phi_i(T) \geqslant 0.$$

It is clear from (2.2.12) that this is true for sufficiently large γ – for instance,

$$\gamma = \sum_{j \neq i} \lambda^*_{ij}[0,\infty)\, \phi_i(T)/\phi_j(T).$$

For such a choice of γ, (2.2.6), (2.2.11) and (2.2.13) show that (2.2.1) holds with $\alpha = \beta + \gamma$, and the theorem is proved.

Theorem I shows, of course, that the local properties of elements of semi-p-matrices follow from those of p-matrices. To take just one example, the fact, for $i \neq j$, that the finite limit

$$q_{ij} = \lim_{t \to 0} p_{ij}(t)/t \tag{2.2.14}$$

exists for any p-matrix (Kingman 1972, theorem 5.5) implies that the same is true for any semi-p-matrix, and therefore, for any M-semigroup.

2.3. *The structure theorem*

Kingman (1973) uses the taming theorem for semi-p-functions to set up a structure theory generalizing that of p-functions. In the same way, theorem I allows a structure theory for semi-p-matrices to be derived from that of p-matrices. To do this, it is convenient to write the Volterra equation (1.2.11) in the equivalent form

$$p(t) = 1 + \int_0^t p(t-s)\, k(s)\, ds, \tag{2.3.1}$$

where $k(t) = -m(t-)$ is left-continuous, increasing and integrable on $(0, 1)$.

Convolutions like that occurring in (2.3.1) will be denoted by $*$, so that the Volterra equation becomes

$$p = 1 + p * k.$$

We shall use without comment the associativity and commutativity of $*$, though remembering that convolutions of matrix-valued functions are not in general commutative.

THEOREM II. *If $\boldsymbol{p}$ is any semi-p-matrix of order N, there exists a unique matrix-valued function*

$$\boldsymbol{k}(t) = (k_{ij}(t);\, i, j = 1, 2, \ldots, N) \quad (t > 0) \tag{2.3.2}$$

such that

(i) *k_{ij} is increasing and left-continuous in* $(0, \infty)$,

(ii) *$k_{ij}(t) \geqslant 0$ for $t > 0$, $i \neq j$,*

(iii) $\int_0^1 |k_{ii}(t)|\,\mathrm{d}t < \infty$, *and*

(iv) *the matrix Volterra equations*

$$\boldsymbol{p}(t) = \boldsymbol{I} + \int_0^t \boldsymbol{k}(s)\,\boldsymbol{p}(t-s)\,\mathrm{d}s, \tag{2.3.3}$$

$$\boldsymbol{p}(t) = \boldsymbol{I} + \int_0^t \boldsymbol{p}(t-s)\,\boldsymbol{k}(s)\,\mathrm{d}s, \tag{2.3.3*}$$

hold for all $t > 0$. Conversely, if $\boldsymbol{k}$ is any function satisfying (i), (ii) *and* (iii), *then* (2.3.3) (*and* (2.3.3*)) *has a unique solution $\boldsymbol{p}$, which is a semi-p-matrix.*

Proof. Let $\boldsymbol{p}$ be a semi-p-matrix, and $T > 0$. Invoke theorem I to choose a number α and a p-matrix $\tilde{\boldsymbol{p}}$ such that

$$\boldsymbol{p}(t) = \tilde{\boldsymbol{p}}(t)\,\mathrm{e}^{\alpha t} \quad (0 \leqslant t \leqslant T), \tag{2.3.4}$$

and use the structure theorem for p-matrices (Kingman 1972, theorem 5.2) to write

$$\int_0^\infty \tilde{\boldsymbol{p}}(t)\,\mathrm{e}^{-\theta t}\,\mathrm{d}t = \boldsymbol{q}(\theta)^{-1} \quad (\theta > 0),$$

$$q_{ii}(\theta) = \theta + \int_{(0,\infty]} (1-\mathrm{e}^{-\theta x})\,\mu_i(\mathrm{d}x),$$

$$q_{ij}(\theta) = -\int_{(0,\infty]} \mathrm{e}^{-\theta x}\,\lambda_{ij}(\mathrm{d}x) \quad (i \neq j),$$

where μ_i is a measure on $(0, \infty]$ with

$$\int_{(0,\infty]} (1-\mathrm{e}^{-x})\,\mu_i(\mathrm{d}x) < \infty,$$

and λ_{ij} a totally finite measure on $[0, \infty)$. Define $\tilde{\boldsymbol{k}}(t)$ by

$$\tilde{k}_{ii}(t) = -\mu_i[t, \infty],$$

$$\tilde{k}_{ij}(t) = \lambda_{ij}[0, t) \quad (i \neq j),$$

and verify that $\tilde{\boldsymbol{k}}$ satisfies (i), (ii) and (iii). For $\theta > 0$,

$$\int_0^\infty \tilde{\boldsymbol{k}}(t)\,\mathrm{e}^{-\theta t}\,\mathrm{d}t = -\theta^{-1}\{\boldsymbol{q}(\theta) - \theta\boldsymbol{I}\},$$

so that

$$\theta^{-1}\boldsymbol{I} = \int_0^\infty \tilde{\boldsymbol{p}}(t)\,\mathrm{e}^{-\theta t}\,\mathrm{d}t \left\{\boldsymbol{I} - \int_0^\infty \tilde{\boldsymbol{k}}(t)\,\mathrm{e}^{-\theta t}\,\mathrm{d}t\right\}$$

$$= \left\{\boldsymbol{I} - \int_0^\infty \tilde{\boldsymbol{k}}(t)\,\mathrm{e}^{-\theta t}\,\mathrm{d}t\right\} \int_0^\infty \tilde{\boldsymbol{p}}(t)\,\mathrm{e}^{-\theta t}\,\mathrm{d}t,$$

and inversion of the Laplace transforms (with the use of the continuity of $\tilde{\boldsymbol{p}}$ and the left-continuity of $\tilde{\boldsymbol{k}}$) shows that (2.3.3) and (2.3.3*) hold with $\boldsymbol{p}$ replaced by $\tilde{\boldsymbol{p}}$ and $\boldsymbol{k}$ by $\tilde{\boldsymbol{k}}$.

Now define

$$\boldsymbol{k}_T(t) = \alpha\boldsymbol{I} + \tilde{\boldsymbol{k}}(t)\,\mathrm{e}^{\alpha t} - \alpha\int_0^t \tilde{\boldsymbol{k}}(s)\,\mathrm{e}^{\alpha s}\,\mathrm{d}s.$$

Then $\boldsymbol{k}_T$ is left-continuous and increasing, since it is easily checked that, for $0 < t_1 < t_2$,

$$\boldsymbol{k}_T(t_2) - \boldsymbol{k}_T(t_1) = \int_{[t_1, t_2)} e^{\alpha t} \, d\tilde{\boldsymbol{k}}(t) \geqslant \mathbf{0}.$$

For $i \neq j$,

$$(\boldsymbol{k}_T(t))_{ij} = \int_{[0,t)} e^{\alpha t} \lambda_{ij}(dt) \geqslant 0,$$

while for $t < 1$,

$$(\boldsymbol{k}_T(1) - \boldsymbol{k}_T(t))_{ii} = \int_{[t,1)} e^{\alpha t} \mu_i(dt),$$

so that $\boldsymbol{k}_T$ satisfies (i), (ii) and (iii). Moreover, for $t \leqslant T$,

$$\begin{aligned}\int_0^t \boldsymbol{k}_T(s)\, \boldsymbol{p}(t-s)\, ds &= \int_0^t \left\{\alpha \boldsymbol{I} + \tilde{\boldsymbol{k}}(s)\, e^{\alpha s} - \alpha \int_0^s \tilde{\boldsymbol{k}}(s-u)\, e^{\alpha(s-u)}\, du\right\} \tilde{\boldsymbol{p}}(t-s)\, e^{\alpha(t-s)}\, ds \\ &= e^{\alpha t}\left\{\alpha \int_0^t \tilde{\boldsymbol{p}}(t-s)\, e^{\alpha s}\, ds + [\tilde{\boldsymbol{p}}(t) - \boldsymbol{I}] - \alpha \int_0^t [\tilde{\boldsymbol{p}}(t-u) - \boldsymbol{I}]\, e^{-\alpha u}\, du\right\} \\ &= \tilde{\boldsymbol{p}}(t)\, e^{\alpha t} - \boldsymbol{I} = \boldsymbol{p}(t) - \boldsymbol{I}.\end{aligned}$$

Hence (2.3.3), and likewise (2.3.3*) is satisfied for $t \leqslant T$ with $\boldsymbol{k}$ replaced by $\boldsymbol{k}_T$.

Now repeat the process with T replaced by a larger value U, to arrive at a function $\boldsymbol{k}_U$ satisfying (2.3.3) in $t \leqslant U$. Then

$$\boldsymbol{p} - \boldsymbol{I} = \boldsymbol{k}_T * \boldsymbol{p} \quad \text{on} \quad [0, T],$$

so that $\boldsymbol{\Delta} = \boldsymbol{k}_U - \boldsymbol{k}_T$ satisfies

$$\boldsymbol{\Delta} * \boldsymbol{p} = 0 \quad \text{on} \quad [0, T].$$

Hence, on $[0, T]$,

$$\begin{aligned}\boldsymbol{\Delta} * \boldsymbol{I} &= \boldsymbol{\Delta} * (\boldsymbol{p} - \boldsymbol{p} * \boldsymbol{k}_T) \\ &= \boldsymbol{\Delta} * \boldsymbol{p} - (\boldsymbol{\Delta} * \boldsymbol{p}) * \boldsymbol{k}_T = \mathbf{0},\end{aligned}$$

so that

$$\int_0^t \boldsymbol{\Delta}(s)\, ds = \mathbf{0} \quad (t \leqslant T).$$

Since $\boldsymbol{\Delta}$ is left-continuous,

$$\boldsymbol{k}_U(t) = \boldsymbol{k}_T(t) \quad (0 < t \leqslant T),$$

and it follows that there exists a function $\boldsymbol{k}$, satisfying (i), (ii) and (iii), whose restriction to $(0, T]$ is $\boldsymbol{k}_T$, for any $T > 0$, and which therefore satisfies (2.3.3) and (2.3.3*) for all $t > 0$. Moreover, the argument just given shows that $\boldsymbol{k}$ is uniquely determined in terms of $\boldsymbol{p}$ by either of the equations (2.3.3) or (2.3.3*).

Conversely, suppose that $\boldsymbol{k}$ satisfies (i), (ii) and (iii), and fix $T > 0$. For $\alpha > 0$, define

$$\boldsymbol{k}_\alpha(t) = -\alpha \boldsymbol{I} + \boldsymbol{k}(t)\, e^{-\alpha t} + \alpha \int_0^t \boldsymbol{k}(s)\, e^{-\alpha s}\, ds,$$

and verify as before that $\boldsymbol{k}_\alpha$ satisfies (i), (ii) and (iii). Choose α so large that

$$(\boldsymbol{k}_\alpha(T))_{ii} \leqslant 0 \quad (i = 1, 2, \ldots, N).$$

Let μ_i be any measure on $(0, \infty]$ such that

$$\mu_i[t, \infty] = -(\boldsymbol{k}_\alpha(t))_{ii} \quad (0 < t \leqslant T),$$

and λ_{ij} $(i \neq j)$ any totally finite measure on $[0, \infty)$ such that

$$\lambda_{ij}[0, t) = (\boldsymbol{k}_\alpha(t))_{ij} \quad (0 \leqslant t \leqslant T).$$

Then these measures satisfy the conditions of the structure theorem for p-matrices, and determine a p-matrix $\boldsymbol{p}_\alpha$. If

$$\boldsymbol{p}^T(t) = \boldsymbol{p}_\alpha(t)\,\mathrm{e}^{\alpha t},$$

then $\boldsymbol{p}^T$ is a semi-p-matrix, and calculations like those just performed show that (2.3.3) and (2.3.3*) hold in $t \leqslant T$ with $\boldsymbol{p}$ replaced by $\boldsymbol{p}^T$.

Repeat the construction with T replaced by a larger value U, to obtain a semi-p-matrix $\boldsymbol{p}^U$ satisfying (2.3.3) and (2.3.3*) in $t \leqslant U$. Then the continuous function

$$\boldsymbol{\Gamma}(t) = \boldsymbol{p}^U(t) - \boldsymbol{p}^T(t)$$

satisfies

$$\boldsymbol{\Gamma}(t) = \int_0^t \boldsymbol{\Gamma}(t-s)\,\boldsymbol{k}(s)\,\mathrm{d}s \quad (0 \leqslant t \leqslant T).$$

Using any matrix norm, choose θ so large that

$$\int_0^T \|\boldsymbol{k}(s)\|\,\mathrm{e}^{-\theta s}\,\mathrm{d}s \leqslant \tfrac{1}{2}.$$

Then

$$\begin{aligned}\int_0^T \|\boldsymbol{\Gamma}(t)\|\,\mathrm{e}^{-\theta t}\,\mathrm{d}t &\leqslant \int_0^T\int_0^t \|\boldsymbol{\Gamma}(t-s)\|\,.\,\|\boldsymbol{k}(s)\|\,\mathrm{d}s\,\mathrm{e}^{-\theta t}\,\mathrm{d}t \\ &\leqslant \int_0^T \|\boldsymbol{\Gamma}(t)\|\,\mathrm{e}^{-\theta t}\,\mathrm{d}t \int_0^T \|\boldsymbol{k}(s)\|\,\mathrm{e}^{-\theta s}\,\mathrm{d}s \\ &\leqslant \frac{1}{2}\int_0^T \|\boldsymbol{\Gamma}(t)\|\,\mathrm{e}^{-\theta t}\,\mathrm{d}t.\end{aligned}$$

Since $\boldsymbol{\Gamma}$ is continuous, this means that $\boldsymbol{\Gamma} = \mathbf{0}$ on $[0, T]$, so that

$$\boldsymbol{p}^U(t) = \boldsymbol{p}^T(t) \quad (0 \leqslant t \leqslant T).$$

Hence there exists a function $\boldsymbol{p}$ such that for all $T > 0$,

$$\boldsymbol{p}(t) = \boldsymbol{p}^T(t) \quad (0 \leqslant t \leqslant T),$$

and $\boldsymbol{p}$ is clearly the unique solution of (2.3.3), and of (2.3.3*), in $t > 0$. Moreover, if

$$0 < t_1 < t_2 < \ldots < t_n,$$

then

$$\boldsymbol{F}(t_1, t_2, \ldots, t_n; \boldsymbol{p}) = \boldsymbol{F}(t_1, t_2, \ldots, t_n; \boldsymbol{p}^{t_n}) \geqslant 0,$$

so that $\boldsymbol{p}$ is a semi-p-matrix, and the theorem is proved.

The theorem sets up a one-to-one correspondence between semi-p-matrices $\boldsymbol{p}$ and functions $\boldsymbol{k}$ satisfying (i), (ii) and (iii), expressed by either of the equations (2.3.3) and (2.3.3*), which will be called the *canonical correspondence*. The proof in fact yields rather more than is asserted. For example, if two semi-p-matrices $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ correspond respectively to $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, then

(*a*) if $\boldsymbol{p}_1(t) = \boldsymbol{p}_2(t)$ for $t \leqslant T$ then $\boldsymbol{\Delta} = \boldsymbol{k}_1 - \boldsymbol{k}_2$

satisfies

$$\boldsymbol{p}_1 * \boldsymbol{\Delta} = \mathbf{0} \quad \text{on} \quad [0, T],$$

and the first uniqueness argument shows that $\boldsymbol{\Delta} = \mathbf{0}$ on $(0, T]$;

(*b*) if $\boldsymbol{k}_1(t) = \boldsymbol{k}_2(t)$ for $t \leqslant T$ then $\boldsymbol{\Gamma} = \boldsymbol{p}_1 - \boldsymbol{p}_2$

satisfies

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma} * \boldsymbol{k}_1 \quad \text{on} \quad (0, T],$$

and the second uniqueness argument shows that $\boldsymbol{\Gamma} = \mathbf{0}$ on $[0, T]$. Thus the canonical correspondence is local, in the sense of the following theorem.

THEOREM III. *Under the conditions of theorem II,*
(*a*) $\boldsymbol{k}(t)$ *is uniquely determined by the values of* $\boldsymbol{p}$ *on* $(0, t)$, *and*
(*b*) $\boldsymbol{p}(t)$ *is uniquely determined by the values of* $\boldsymbol{k}$ *on* $(0, t)$.

When $i \neq j$, k_{ij} is positive and increasing, so that the finite positive limit

$$k_{ij}(0+) = \lim_{t\to 0} k_{ij}(t)$$

exists. By (2.3.3*),

$$p_{ij}(t) = \int_0^t \sum_{l=1}^{N} p_{il}(t-s)\, k_{lj}(s)\, \mathrm{d}s,$$

and it follows from (2.1.10) that

$$p_{ij}(t) = tk_{ij}(0+) + o(t).$$

Hence the Doob–Kolmogorov limits (2.2.14) are given by

$$q_{ij} = k_{ij}(0+) \quad (i \neq j). \tag{2.3.5}$$

The same result holds when $i = i$, except that the derivative of p_{ii} at 0 and the limit $k_{ii}(0)$ may equal $-\infty$.

2.4. *Decompositions*

In the theory of Markov chains, an important part is played by the first passage and last exit decompositions

$$p_{ij} = f_{ij} * p_{jj} = p_{ii} * g_{ij} \quad (i \neq j), \tag{2.4.1}$$

where f_{ij} and g_{ij} are non-negative and locally integrable. These are usually proved by a 'skeleton' argument (cf. Chung 1967), but Kingman (1965) pointed out that they are simple consequences of the fact that, in a Markov semigroup, the matrix

$$\begin{pmatrix} p_{ii} & p_{ij} \\ p_{ji} & p_{jj} \end{pmatrix} \tag{2.4.2}$$

is a p-matrix. In fact, if $\boldsymbol{p}$ is any (2×2) p-matrix, then (Kingman 1972, theorem 5.3)

$$p_{12} = f_{12} * p_{22} = p_{11} * g_{12}, \tag{2.4.3}$$

where, in our present notation,

$$f_{12} = p_1 * \mathrm{d}k_{12}, \quad g_{12} = \mathrm{d}k_{12} * p_2, \tag{2.4.4}$$

and for $\alpha = 1, 2$, p_α is the p-function with canonical measure $\mathrm{d}k_{\alpha\alpha}$, and therefore satisfying

$$p_\alpha = 1 + p_\alpha * k_{\alpha\alpha}. \tag{2.4.5}$$

In (2.4.4) the asterisk denotes Stieltjes convolution:

$$(p * \mathrm{d}k)\,(t) = \int_{[0,t)} p(t-s)\, \mathrm{d}k(s),$$

with the convention that $k(0) = 0$; note that if k is left-continuous,

$$1 * \mathrm{d}k = k.$$

The fact that the elements f and g of the decompositions (2.4.3) themselves admit further decompositions (2.4.4) is a crucial step in the solution of the Markov characterization problem. It is not difficult to use theorem II to extend this result from p-matrices to semi-p-matrices, but for our present purposes a rather more general result is needed.

THEOREM IV. *Let* $\boldsymbol{p}$ *be a semi-p-matrix of order* $(m+n)$, $\boldsymbol{k}$ *the matrix to which it corresponds,* $\boldsymbol{k}^0$ *the submatrix of* $\boldsymbol{k}$ *consisting of the first m rows and columns, and* $\boldsymbol{p}^0$ *the semi-p-matrix of order m corresponding to* $\boldsymbol{k}^0$. *Then, for* $i \leqslant m < j$,

$$p_{ij} = \sum_{\alpha=1}^{m} \sum_{\beta=m+1}^{m+n} p_{i\alpha}^0 * \mathrm{d}k_{\alpha\beta} * p_{\beta j}. \tag{2.4.6}$$

Proof. Equation (2.3.3) shows that, for $i \leqslant m < j$,

$$p_{ij} = \sum_{\beta=1}^{m} (k_{i\beta} * p_{\beta j}) + \sum_{\beta=m+1}^{m+n} (k_{i\beta} * p_{\beta j})$$

$$= \sum_{\beta=1}^{m} (k_{i\beta}^0 * p_{\beta j}) + \psi_{ij} \quad \text{(say)}.$$

Hence

$$\sum_{\alpha=1}^{m} p_{i\alpha}^0 * p_{\alpha j} = \sum_{\alpha,\beta=1}^{m} (p_{i\alpha}^0 * k_{\alpha\beta}^0 * p_{\beta j}) + \sum_{\alpha=1}^{m} p_{i\alpha}^0 * \psi_{\alpha j}$$

$$= \sum_{\beta=1}^{m} (p_{i\beta}^0 - \delta_{i\beta}) * p_{\beta j} + \sum_{\alpha=1}^{m} p_{i\alpha}^0 * \psi_{\alpha j},$$

and therefore

$$1 * p_{ij} = \sum_{\alpha=1}^{m} p_{i\alpha}^0 * \psi_{\alpha j}$$

$$= \sum_{\alpha=1}^{m} \sum_{\beta=m+1}^{m+n} p_{i\alpha}^0 * 1 * \mathrm{d}k_{\alpha\beta} * p_{\beta j}$$

$$= 1 * \sum_{\alpha=1}^{m} \sum_{\beta=m+1}^{m+n} p_{i\alpha}^0 * \mathrm{d}k_{\alpha\beta} * p_{\beta j}.$$

This is the integrated form of (2.4.6), which follows by continuity. Hence the proof is complete.

Taking the transpose of $\boldsymbol{p}$, and interchanging m and n by the obvious re-labelling, we have the dual form of (2.4.6):

$$p_{ij} = \sum_{\alpha=1}^{m} \sum_{\beta=m+1}^{m+n} p_{i\alpha} * \mathrm{d}k_{\alpha\beta} * p_{\beta j}^1, \tag{2.4.7}$$

where $\boldsymbol{p}^1$ corresponds to the submatrix $\boldsymbol{k}^1$ consisting of the last n rows and columns of $\boldsymbol{k}$. Clearly (2.4.3) and (2.4.4) follow on setting $m = n = 1$.

In particular, applying the result to (2.4.2) we recover Jurkat's result that (2.4.1) holds in any M-semigroup, together with the refinement of the further decompositions of f_{ij} and g_{ij}.

There is another consequence of (2.3.3) which will be needed later. For $s, t > 0$, the function

$$\boldsymbol{K}(s,t) = \int_0^t \{\boldsymbol{k}(s+x) - \boldsymbol{k}(x)\}\,\mathrm{d}x \tag{2.4.8}$$

is finite and continuous, and satisfies

$$\boldsymbol{K}(s_2,t_2) - \boldsymbol{K}(s_1,t_2) - \boldsymbol{K}(s_2,t_1) + \boldsymbol{K}(s_1,t_1) \geqslant \boldsymbol{0}$$

whenever $s_1 < s_2$, $t_1 < t_2$. Hence there is a positive matrix-valued measure $\boldsymbol{\kappa}$ on the positive quadrant $(0,\infty)^2$, finite on bounded sets, and defined uniquely by the requirement that

$$\boldsymbol{\kappa}((0,s) \times (0,t)) = \boldsymbol{K}(s,t) \tag{2.4.9}$$

for all $s, t > 0$. (It may make this construction less mysterious to remark that, if $\boldsymbol{k}$ is absolutely continuous with derivative $\boldsymbol{h}$, then $\boldsymbol{\kappa}$ has a density

$$\boldsymbol{\kappa}(\mathrm{d}s\,\mathrm{d}t) = \boldsymbol{h}(s+t)\,\mathrm{d}s\,\mathrm{d}t.)$$

From (2.4.8) and (2.4.9),

$$\int_0^s\int_0^t \boldsymbol{\kappa}(\mathrm{d}u\,\mathrm{d}v)\,\boldsymbol{p}(t-v) = \int_0^t \{\boldsymbol{k}(s+x)-\boldsymbol{k}(x)\}\,\boldsymbol{p}(t-x)\,\mathrm{d}x$$

$$= \boldsymbol{p}(s+t)-\boldsymbol{p}(t)-\int_{-s}^0 \boldsymbol{k}(s+x)\,\boldsymbol{p}(t-x)\,\mathrm{d}x$$

$$= \boldsymbol{p}(s+t)-\boldsymbol{p}(s)\,\boldsymbol{p}(t)-\int_0^s \boldsymbol{k}(u)\,\{\boldsymbol{p}(s-u+t)-\boldsymbol{p}(s-u)\,\boldsymbol{p}(t)\}\,\mathrm{d}u,$$

using (2.3.3). Holding t fixed and writing

$$\boldsymbol{A}(s) = \int_0^s\int_0^t \boldsymbol{\kappa}(\mathrm{d}u\,\mathrm{d}v)\,\boldsymbol{p}(t-v), \quad \boldsymbol{B}(s) = \boldsymbol{p}(s+t)-\boldsymbol{p}(s)\,\boldsymbol{p}(t),$$

we have therefore

$$\boldsymbol{A} = \boldsymbol{B}-\boldsymbol{k}*\boldsymbol{B},$$

whence by (2.3.3*),

$$\boldsymbol{I}*\boldsymbol{p}*\mathrm{d}\boldsymbol{A} = \boldsymbol{p}*\boldsymbol{A} = \boldsymbol{p}*\boldsymbol{B}-(\boldsymbol{p}*\boldsymbol{k})*\boldsymbol{B} = \boldsymbol{I}*\boldsymbol{B},$$

so that

$$\boldsymbol{B} = \boldsymbol{p}*\mathrm{d}\boldsymbol{A}.$$

Hence we have proved that, for any semi-p-matrix,

$$\boldsymbol{p}(s+t)-\boldsymbol{p}(s)\,\boldsymbol{p}(t) = \int_0^s\int_0^t \boldsymbol{p}(s-u)\,\boldsymbol{\kappa}(\mathrm{d}u\,\mathrm{d}v)\,\boldsymbol{p}(t-v), \tag{2.4.10}$$

where $\boldsymbol{\kappa}$ is defined in terms of the matrix $\boldsymbol{k}$ corresponding to $\boldsymbol{p}$ by (2.4.9) and (2.4.8).

3. Applications to M-semigroups

The rest of this paper will be devoted to applying the theory of semi-p-matrices to the determination of properties of the elements, and more generally the finite submatrices, of M-semigroups. To this end, it will be assumed throughout that

$$\boldsymbol{P}(t) = (p_{ij}(t); i, j \in I)$$

satisfies (1.1.1) and (1.1.3). If J is any finite subset of I, $\boldsymbol{p}^J$ will devote the principal submatrix

$$\boldsymbol{p}^J(t) = (p_{ij}(t); i, j \in J).$$

Since $\boldsymbol{p}^J$ is a semi-p-matrix, theorem II establishes the existence of the corresponding matrix

$$\boldsymbol{k}^J(t) = (k_{ij}^J(t); i, j \in J)$$

such that $\boldsymbol{p}^J$ and $\boldsymbol{k}^J$ satisfy (2.3.3) and (2.3.3*).

3.1. *Taboo semigroups*

If H is any subset of I, there is an important semigroup with index set $(I-H)$ which is obtained from $\boldsymbol{P}$ by 'forbidding transitions into H'. In the Markov case it is usual to define this *taboo semigroup* either probabilistically (i.e. by considering certain measures on the space of functions from $[0, \infty)$ into I) or by a slightly delicate limiting process. However, theorem II makes possible a more direct and convenient definition, at least when H is finite. (When H is infinite, any definition must take note of the intrinsic topology which $\boldsymbol{P}$ defines on I (Neveu 1961).)

Thus let H be any finite subset of I. For any finite subset J disjoint from H, write

$${}_H\boldsymbol{k}^J = (k_{ij}^{J\cup H}; i, j \in J) \tag{3.1.1}$$

for the principal submatrix of $\boldsymbol{k}^{J\cup H}$ corresponding to J. Then the second part of theorem II shows that, in the canonical correspondence, ${}_H\boldsymbol{k}^J$ corresponds to a unique semi-p-matrix

$${}_H\boldsymbol{p}^J = ({}_Hp_{ij}^J;\ i,j\in J).$$

THEOREM V. *For any $i,j\notin H$, the function ${}_Hp_{ij} = {}_Hp_{ij}^J$ does not depend on J so long as*

$$\{i,j\}\subseteq J\subseteq I-H.$$

Moreover,
$$ {}_H\boldsymbol{P} = ({}_Hp_{ij};\ i,j\in I-H) \tag{3.1.2}$$
is an M-semigroup with index set $(I-H)$.

Proof. Let $r\notin J\cup H$, $K = J\cup\{r\}$ and $\boldsymbol{k} = \boldsymbol{k}^{K\cup H}$. Then

$$\boldsymbol{p}^{K\cup H} = \boldsymbol{I}+\boldsymbol{p}^{K\cup H}*\boldsymbol{k},$$

so that, for $i,j\in J\cup H$,
$$p_{ij} = \delta_{ij}+\sum_{\alpha\in J\cup H}(p_{i\alpha}*k_{\alpha j})+(p_{ir}*k_{rj}).$$

Moreover, (2.4.7) shows that
$$p_{ir} = \sum_{\alpha\in J\cup H} p_{i\alpha}*\mathrm{d}k_{\alpha r}*p_r,$$

where p_r is the semi-p-function satisfying

$$p_r = 1+p_r*k_{rr}.$$

Hence
$$p_{ij} = \delta_{ij}+\sum_{\alpha\in J\cup H} p_{i\alpha}*(k_{\alpha j}+\mathrm{d}k_{\alpha r}*p_r*k_{rj})$$

for $i,j\in J\cup H$, so that
$$k_{ij}^{J\cup H} = k_{ij}+\mathrm{d}k_{ir}*p_r*k_{rj}.$$

In particular, restricting i and j to J, we have

$${}_Hk_{ij}^J = {}_Hk_{ij}^K+\mathrm{d}\,{}_Hk_{ir}^K*p_r*{}_Hk_{rj}^K, \tag{3.1.3}$$

where
$$p_r = 1+p_r*{}_Hk_{rr}^K.$$

Exactly similar calculations show that the submatrix

$$({}_Hp_{ij}^K;\ i,j\in J)$$

satisfies (2.3.3) with $\boldsymbol{k}$ given by the right-hand side of (3.1.3), and therefore

$${}_Hp_{ij}^J = {}_Hp_{ij}^K\quad (i,j\in J), \tag{3.1.4}$$

where $K = J\cup\{r\}$. Repeated application of (3.1.4) shows that the same equation holds for any finite K with $J\subseteq K\subseteq I-H$. Hence, if J_1 and J_2 are finite subsets of $I-H$, with $\{i,j\}\subseteq J_1\cap J_2$, then

$${}_Hp_{ij}^{J_1} = {}_Hp_{ij}^{J_1\cup J_2} = {}_Hp_{ij}^{J_2},$$

and the first part of the theorem is proved.

Notice that we have established that, for any finite J, K with $J\subseteq K\subseteq I-H$, there is a commutative diagram

$$\begin{array}{ccccccc} \boldsymbol{p}^{K\cup H} & \leftrightarrow & \boldsymbol{k}^{K\cup H} & \rightarrow & {}_H\boldsymbol{k}^K & \leftrightarrow & {}_H\boldsymbol{p}^K \\ \downarrow & & & & & & \downarrow \\ \boldsymbol{p}^{J\cup H} & \leftrightarrow & \boldsymbol{k}^{J\cup H} & \rightarrow & {}_H\boldsymbol{k}^J & \leftrightarrow & {}_H\boldsymbol{p}^J \end{array} \tag{3.1.5}$$

in which the double-ended arrows stand for the canonical correspondence and the single-ended arrows indicate the taking of appropriate submatrices. Moreover, the proof of (3.1.5) depends only on the fact that $\boldsymbol{p}^{K\cup H}$ is a semi-p-matrix.

To complete the proof of the theorem, we have to show that ${}_HP$, now well-defined by (3.1.2), satisfies (1.1.1) on $(I-H)$. For any finite J with $\{i,j\} \subseteq J \subseteq I-H$, the fact that ${}_Hp^J$ is a semi-p-matrix means, by (1.2.1) that

$$\begin{aligned} {}_Hp_{ij}(s+t) &= {}_Hp^J_{ij}(s+t) \geqslant \sum_{\alpha\in J} {}_Hp^J_{i\alpha}(s)\, {}_Hp^J_{\alpha j}(t) \\ &= \sum_{\alpha\in J} {}_Hp_{i\alpha}(s)\, {}_Hp_{\alpha j}(t). \end{aligned}$$

Since J is arbitrary,
$$ {}_Hp_{ij}(s+t) \geqslant \sum_{\alpha\in I-H} {}_Hp_{i\alpha}(s)\, {}_Hp_{\alpha j}(t). \tag{3.1.6}$$

To prove the reverse inequality, let J be again a finite set disjoint from H, and use (2.3.3) to give, for $i, j \in J$,
$$\begin{aligned} p_{ij} &= \delta_{ij} + \sum_{\alpha\in J\cup H} k^{J\cup H}_{i\alpha} * p_{\alpha j} \\ &\geqslant \delta_{ij} + \sum_{\alpha\in J} k^{J\cup H}_{i\alpha} * p_{\alpha j}. \end{aligned}$$

Hence
$$\boldsymbol{p}^J \geqslant \boldsymbol{I} + {}_H\boldsymbol{k}^J * \boldsymbol{p}^J \tag{3.1.7}$$

and (2.3.3*) implies that
$$ {}_H\boldsymbol{p}^J * \boldsymbol{p}^J \geqslant {}_H\boldsymbol{p}^J * \boldsymbol{I} + ({}_H\boldsymbol{p}^J - \boldsymbol{I}) * \boldsymbol{p}^J,$$

so that
$$\int_0^t {}_H\boldsymbol{p}^J(s)\,\mathrm{d}s \leqslant \int_0^t \boldsymbol{p}^J(s)\,\mathrm{d}s$$

for all $t > 0$. Since ${}_H\boldsymbol{k}^J$ is increasing, this implies that
$$ {}_H\boldsymbol{k}^J * {}_H\boldsymbol{p}^J \leqslant {}_H\boldsymbol{k}^J * \boldsymbol{p}^J,$$

which substituted in (3.1.7) gives
$$\boldsymbol{p}^J \geqslant \boldsymbol{I} + {}_H\boldsymbol{k}^J * {}_H\boldsymbol{p}^J = {}_H\boldsymbol{p}^J.$$

Therefore, choosing J to contain i and j, we have the important inequality
$$ {}_Hp_{ij} \leqslant p_{ij} \quad (i, j \notin H). \tag{3.1.8}$$

Now apply (2.4.10) to the semi-p-matrices ${}_H\boldsymbol{p}^J$ and $\boldsymbol{p}^{J\cup H}$ to give, for $i, j \in J \subseteq I-H$,
$$\begin{aligned} {}_Hp_{ij}(s+t) - \sum_{\alpha\in J} {}_Hp_{i\alpha}(s)\, {}_Hp_{\alpha j}(t) &= \int_0^s\int_0^t \sum_{\alpha,\beta\in J} {}_Hp_{i\alpha}(s-u)\, {}_H\kappa^J_{\alpha\beta}(\mathrm{d}u\,\mathrm{d}v)\, {}_Hp_{\alpha j}(t-v) \\ &\leqslant \int_0^s\int_0^t \sum_{\alpha,\beta\in J} p_{i\alpha}(s-u)\, \kappa^{J\cup H}_{\alpha\beta}(\mathrm{d}u\,\mathrm{d}v)\, p_{\alpha j}(t-v) \\ &\leqslant \int_0^s\int_0^t \sum_{\alpha,\beta\in J\cup H} p_{i\alpha}(s-u)\, \kappa^{J\cup H}_{\alpha\beta}(\mathrm{d}u\,\mathrm{d}v)\, p_{\alpha j}(t-v) \\ &= p_{ij}(s+t) - \sum_{\alpha\in J\cup H} p_{i\alpha}(s)\, p_{\alpha j}(t). \end{aligned}$$

Hence
$$p_{ij}(s+t) - {}_Hp_{ij}(s+t) \geqslant \sum_{\alpha\in J} \{p_{i\alpha}(s)\, p_{\alpha j}(t) - {}_Hp_{i\alpha}(s)\, {}_Hp_{\alpha j}(t)\} + \sum_{\alpha\in H} p_{i\alpha}(s)\, p_{\alpha j}(t).$$

Since J is arbitrary, and the expression in brackets is non-negative, J may be replaced by $I-H$, to give
$$\begin{aligned} p_{ij}(s+t) - {}_Hp_{ij}(s+t) &\geqslant -\sum_{\alpha\in I-H} {}_Hp_{i\alpha}(s)\, {}_Hp_{\alpha j}(t) + \sum_{\alpha\in I} p_{i\alpha}(s)\, p_{\alpha j}(t) \\ &= -\sum_{\alpha\in I-H} {}_Hp_{i\alpha}(s)\, {}_Hp_{\alpha j}(t) + p_{ij}(s+t). \end{aligned}$$

This is the reverse inequality to (3.1.6), showing that ${}_H\boldsymbol{p}$ satisfies (1.1.1), and completing the proof of the theorem.

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Theorem V may be taken as the starting-point for a systematic theory of taboo semigroups which closely parallels the classical one, and which in turn leads to the concept of dominance as a relation between M-semigroups (Neveu 1961). This line of argument will not be pursued here; for the present purposes the most important consequence of theorem V is a subtler form of the Lévy dichotomy.

Corollary. Each of the non-diagonal elements of the matrix $\boldsymbol{k}^J$ *derived from an M-semigroup is either always or never zero in* $(0, \infty)$.

Proof. If i and j are distinct elements of the finite set J, write $H = J-\{i, j\}$, and apply Chung's form of the Lévy dichotomy (Chung 1963, theorem 10.1) to the M-semigroup ${}_H\boldsymbol{p}$. This shows that ${}_Hp_{ij}$ is either always or never zero. By the construction of the taboo semigroup ${}_H\boldsymbol{P}$, the semi-p-matrix

$$ {}_H\boldsymbol{p}^{\{i,j\}} = \begin{pmatrix} {}_Hp_{ii} & {}_Hp_{ij} \\ {}_Hp_{ji} & {}_Hp_{jj} \end{pmatrix} $$

corresponds to

$$ {}_H\boldsymbol{k}^{\{i,j\}} = \begin{pmatrix} k^J_{ii} & k^J_{ij} \\ k^J_{ji} & k^J_{jj} \end{pmatrix}. $$

Hence, by (2.4.6)

$$ {}_Hp_{ij} = {}_{H\cup\{j\}}p_{ii} * \mathrm{d}k^J_{ij} * {}_Hp_{jj}. $$

Since the first and third elements on the right-hand side are semi-p-functions, they are strictly positive. Hence if ${}_Hp_{ij}(t)$ vanishes for some $t > 0$, it vanishes for all $t > 0$, and k^J_{ij} is identically zero. On the other hand, if ${}_Hp_{ij}(t) > 0$, then $k^J_{ij}(t) > 0$ (since k^J_{ij} is increasing) so that the strict positivity of ${}_Hp_{ij}(t)$ for some, and then for all, $t > 0$ entails the same for k^J_{ij}.

3.2. *Properties of submatrices of M-semigroups*

The corollary to theorem V shows that the submatrix $\boldsymbol{p}^J$ of the M-semigroup $\boldsymbol{P}$ has at least one property not enjoyed by all semi-p-matrices, and that this property is best expressed in terms of the corresponding $\boldsymbol{k}^J$. In this section further such properties are established; these will later be seen to be characteristic of semi-p-matrices derived from M-semigroups.

In this section $\boldsymbol{P}$ will be a fixed M-semigroup with index set I, J will be a fixed finite subset of I, and

$$ \boldsymbol{p} = \boldsymbol{p}^J, \quad \boldsymbol{k} = \boldsymbol{k}^J. $$

THEOREM VI. *There exists a matrix* $\boldsymbol{h}$, *whose elements* h_{ij} $(i, j \in J)$ *are lower semicontinuous functions on* $(0, \infty)$, *such that*

$$ \boldsymbol{k}(t) - \boldsymbol{k}(s) = \int_s^t \boldsymbol{h}(x)\, \mathrm{d}x \quad (0 < s < t < \infty). \tag{3.2.1} $$

Moreover, any element h_{ij} *which is not identically zero satisfies*

$$ h_{ij}(x) > 0 \quad (x > 0), \qquad h_{ij}(x) \geqslant \mathrm{e}^{-\beta x} \quad (x \geqslant 1) \tag{3.2.2} $$

for some constant β.

Proof. For $r \notin J$, apply theorem IV to the semi-p-matrix $\boldsymbol{p}^{J\cup\{r\}}$ to give, for $i, j \in J$,

$$ p_{ir} = \sum_{\alpha\in J} p_{i\alpha} * \mathrm{d}k^r_{\alpha r} * p_r, \quad p_{rj} = \sum_{\beta\in J} p_r * \mathrm{d}k^r_{r\beta} * p_{\beta j}, \tag{3.2.3} $$

when $\boldsymbol{k}^r = \boldsymbol{k}^{J\cup\{r\}}$, $p_r = {}_Jp_{rr}$. Then

$$ \begin{aligned} p_{ij}(s+t) - \sum_{\alpha\in J} p_{i\alpha}(s)\, p_{\alpha j}(t) &= \sum_{r\notin J} p_{ir}(s)\, p_{rj}(t) \\ &= \sum_{r\notin J} (p_{i\alpha} * \mathrm{d}k^r_{\alpha r} * p_r)(s)\, (p_r * \mathrm{d}k^r_{r\beta} * p_{\beta j})(t), \end{aligned} $$

so that $$\boldsymbol{p}(s+t)-\boldsymbol{p}(s)\,\boldsymbol{p}(t) = \int_0^s\int_0^t \boldsymbol{p}(s-u)\,\boldsymbol{H}(u,v)\,\boldsymbol{p}(t-v)\,\mathrm{d}u\,\mathrm{d}v, \tag{3.2.4}$$

where $$H_{ij}(u,v) = \sum_{r\notin J}(\mathrm{d}k_{ir}^r * p_r)(u)\,(p_r * \mathrm{d}k_{rj}^r)(v). \tag{3.2.5}$$

Comparing (3.2.4) with (2.4.10), we have

$$\int_0^s\int_0^t \boldsymbol{p}(s-u)\,\boldsymbol{H}(u,v)\,\boldsymbol{p}(t-v)\,\mathrm{d}u\,\mathrm{d}v = \int_0^s\int_0^t \boldsymbol{p}(s-u)\,\boldsymbol{\kappa}(\mathrm{d}u\,\mathrm{d}v)\,\boldsymbol{p}(t-v).$$

Hence, with t fixed, the function

$$\boldsymbol{\phi}(s) = \int_0^s\int_0^t \boldsymbol{H}(u,v)\,\boldsymbol{p}(t-v)\,\mathrm{d}u\,\mathrm{d}v - \int_0^s\int_0^t \boldsymbol{\kappa}(\mathrm{d}u\,\mathrm{d}v)\,\boldsymbol{p}(t-v)\,\mathrm{d}v$$

satisfies $\boldsymbol{p} * \mathrm{d}\boldsymbol{\phi} = \mathbf{0}$, whence

$$\boldsymbol{\phi} = \boldsymbol{I} * \mathrm{d}\boldsymbol{\phi} = (\boldsymbol{p}-\boldsymbol{k}*\boldsymbol{p}) * \mathrm{d}\boldsymbol{\phi} = \mathbf{0}$$

and therefore $$\int_0^s\int_0^t \boldsymbol{H}(u,v)\,\boldsymbol{p}(t-v)\,\mathrm{d}u\,\mathrm{d}v = \int_0^s\int_0^t \boldsymbol{\kappa}(\mathrm{d}u\,\mathrm{d}v)\,\boldsymbol{p}(t-v)\,\mathrm{d}v.$$

Repeating the argument with s fixed,

$$\int_0^s\int_0^t \boldsymbol{H}(u,v)\,\mathrm{d}u\,\mathrm{d}v = \int_0^s\int_0^t \boldsymbol{\kappa}(\mathrm{d}u\,\mathrm{d}v) = \int_0^t \{\boldsymbol{k}(s+x)-\boldsymbol{k}(x)\}\,\mathrm{d}x.$$

Hence $$\int_0^s \boldsymbol{H}(u,v)\,\mathrm{d}u = \boldsymbol{k}(s+v)-\boldsymbol{k}(v)$$

for all s and almost all v. Since $\boldsymbol{H}$ is clearly measurable this holds for almost all (s,v). Therefore $\boldsymbol{k}$ is absolutely continuous, and there exists $\boldsymbol{h} \geqslant 0$ satisfying (3.2.1). Hence

$$\int_0^s \boldsymbol{H}(u,v)\,\mathrm{d}u = \int_0^s \boldsymbol{h}(u+v)\,\mathrm{d}u$$

for almost all (s,v), so that $$\boldsymbol{H}(u+v) = \boldsymbol{h}(u+v)$$

for almost all (u,v). In particular, for almost all t,

$$t\boldsymbol{h}(t) = \int_0^t \boldsymbol{H}(u,t-u)\,\mathrm{d}u$$
$$= \Big(\sum_{r\in J}(\mathrm{d}k_{ir}^r * p_r * p_r * \mathrm{d}k_{rj}^r);\ i,j\in J\Big).$$

Since we may vary $\boldsymbol{h}$ on a null set without affecting (3.2.1), we can and will define h by

$$h_{ij}(t) = t^{-1}\sum_{r\in J}(\mathrm{d}k_{ir}^r * p_r * p_r * \mathrm{d}k_{rj}^r)(t) \tag{3.2.6}$$

in the confidence that (3.2.1) holds.

That h_{ij} is lower semicontinuous now follows at once from Fatou's lemma. To complete the proof, suppose that, for a particular pair i, j (equal or unequal), h_{ij} is not identically zero. Choose $r \notin J$ so that

$$\mathrm{d}k_{ir}^r * p_r * p_r * \mathrm{d}k_{rj}^r$$

is not identically zero. Then the functions k_{ir}^r and k_{rj}^r are not identically zero, and the corollary to theorem V shows that they are strictly positive on $(0,\infty)$, so that the increasing function

$$l = k_{ir}^r * \mathrm{d}k_{rj}^r$$

is strictly positive on $(0,\infty)$. Moreover, since p_r is a semi-p-function, $\pi = p_r * p_r$ is strictly positive on $(0,\infty)$, so that

$$h_{ij}(t) \geqslant t^{-1}\int_{[0,t)} \pi(t-s)\,\mathrm{d}l(s) > 0 \tag{3.2.7}$$

for all $t > 0$. Again, since p_r is a semi-p-function,

$$p_r(s+t) \geqslant p_r(s)\,p_r(t),$$

from which it follows easily that, for some $\gamma \geqslant 0$,

$$p_r(t) \geqslant \tfrac{1}{2}\,\mathrm{e}^{-\gamma t} \quad (t > 0).$$

Hence $$\pi_r(t) \geqslant \tfrac{1}{4}t\,\mathrm{e}^{-\gamma t},$$
and (3.2.7) implies that, for $t \geqslant 1$,

$$h_{ij}(t) \geqslant t^{-1}\int_{[0,t)} \tfrac{1}{4}(t-s)\,\mathrm{e}^{-\gamma(t-s)}\,\mathrm{d}l(s) \geqslant \tfrac{1}{8}l(\tfrac{1}{2})\,\mathrm{e}^{-\gamma t} \geqslant \mathrm{e}^{-\beta t}$$

for a suitable choice of β. Hence the theorem is proved.

3.3. *Property M*

We are now at last in a position to answer the question posed in § 1.2; which functions p: $(0,\infty) \to (0,\infty)$ have property M, the property of being expressible as a diagonal element p_{ii} in some M-semigroup?

THEOREM VII. *A function p has property M if and only if it satisfies the Volterra equation*

$$p(t) = 1 + \int_0^t p(t-s)\,k(s)\,\mathrm{d}s \tag{3.3.1}$$

for some function k which is either constant or satisfies

$$k(t) - k(s) = \int_s^t h(x)\,\mathrm{d}x, \tag{3.3.2}$$

where

(i) *h is strictly positive (but may equal* $+\infty$*) and lower semicontinuous in* $(0,\infty)$,

(ii) $h(x) \geqslant \mathrm{e}^{-\beta x}$ $(x \geqslant 1)$ *for some constant* β, *and*

(iii) $\int_0^t xh(x)\,\mathrm{d}x < \infty$ *for all* $t > 0$.

Proof. The necessity of these conditions is just the assertion of theorem VI when $N = 1$. To prove the sufficiency, note first that, if k is a constant, then $p(t) = \mathrm{e}^{kt}$ trivially has property M. Excluding this trivial case, assume that p satisfies (3.3.1), where k satisfies (3.3.2) and h satisfies (i), (ii) and (iii). Since (iii) implies that h is finite almost everywhere, we may choose ξ such that $h(\xi) < \infty$.

The functions $$H_n(t) = \inf_{0<x<\infty}\{h(x) + n|x-t|\}$$
are continuous and satisfy

$$0 < H_1(t) \leqslant H_2(t) \leqslant \ldots \leqslant H_n(t) \to h(t)$$

and $$H_n(t) \leqslant h(\xi) + n|\xi - t|$$

(cf. Kingman 1972, p. 132). Hence the continuous functions

$$h_1(t) = H_1(t)\,\mathrm{e}^{-t},$$
$$h_n(t) = H_n(t)\,\mathrm{e}^{-t/n} - H_{n-1}(t)\,\mathrm{e}^{-t/(n-1)} \quad (n \geqslant 2),$$

satisfy
$$h(t) = \sum_{n=1}^{\infty} h_n(t), \tag{3.3.3}$$
$$\int_0^{\infty} h_n(t)\,\mathrm{d}t < \infty, \tag{3.3.4}$$
and
$$h_n(t) \geqslant H_n(t)\,\mathrm{e}^{-t/n}\,(1-\mathrm{e}^{-t/n(n-1)}).$$

Hence (using the easily verified fact that (ii) implies that $H_1(t) \geqslant \mathrm{e}^{-\beta t}$ for sufficiently large t) h_n satisfies all the conditions assumed of h, together with (3.3.4). Define
$$k_n(t) = k(1)\,2^{-n} + \int_1^t h_n(x)\,\mathrm{d}x;$$
then k_n is continuous, bounded and increasing, and
$$k = \sum_{n=1}^{\infty} k_n. \tag{3.3.5}$$

Let p_n be the semi-p-function satisfying (3.3.1) with k replaced by k_n. Then, for $\alpha > 0$, the semi-p-function $p_n(t)\,\mathrm{e}^{-\alpha t}$ corresponds to the function
$$\begin{aligned} k_{n\alpha}(t) &= -\alpha + k_n(t)\,\mathrm{e}^{-\alpha t} + \alpha\int_0^t k_n(s)\,\mathrm{e}^{-\alpha s}\,\mathrm{d}s \\ &= -\alpha + k_n(0) + \int_0^t h_n(s)\,\mathrm{e}^{-\alpha s}\,\mathrm{d}s. \end{aligned}$$

Choose $\alpha = \alpha_n$ so large that $\quad k_{n\alpha}(t) \leqslant 0 \quad$ for all $\quad t > 0.$

Then
$$p_n(t)\,\mathrm{e}^{-\alpha_n t}$$
is a p-function, and its canonical measure has density
$$h_n(t)\,\mathrm{e}^{-\alpha_n t}.$$

Hence the Markov characterization theorem (Kingman 1972, theorem 6.1) shows that there is a Markov semigroup $\boldsymbol{P}^{[n]}$ (whose index set may be taken to consist of the non-negative integers) such that
$$p_{00}^{[n]}(t) = p_n(t)\,\mathrm{e}^{-\alpha_n t}.$$

Then
$$\boldsymbol{P}^{(n)}(t) = \boldsymbol{P}^{[n]}(t)\,\mathrm{e}^{\alpha_n t}$$
is an M-semigroup with
$$p_{00}^{(n)}(t) = p_n(t). \tag{3.3.6}$$

Let I consist of 0 and the pairs (α, n) $(\alpha, n = 1, 2, 3, \ldots)$. Define functions p_{ij} $(i, j \in I)$ by
$$\left.\begin{aligned} p_{00} &= p, \\ p_{0,(\beta,n)} &= p * g_{0\beta}^{(n)}, \\ p_{(\alpha,m),0} &= f_{\alpha 0}^{(m)} * p, \\ p_{(\alpha,m),(\beta,n)} &= \delta_{mn}\,p_{\alpha\beta}^{(m)} + f_{\alpha 0}^{(m)} * p * g_{0\beta}^{(n)}, \end{aligned}\right\} \tag{3.3.7}$$
where $f_{\alpha 0}^{(m)}$ and $g_{0\beta}^{(n)}$ are respectively the functions satisfying
$$p_{\alpha 0}^{(m)} = f_{\alpha 0}^{(m)} * p_{00}^{(m)}, \quad p_{0\alpha}^{(n)} = p_{00}^{(n)} * g_{0\beta}^{(n)}.$$

The theorem will be proved if it can be shown that the functions p_{ij} satisfy (1.1.1), for then (1.1.3) is clearly satisfied, and (p_{ij}) is an M-semigroup with $p_{00} = p$. We shall exhibit only one of the four different calculations needed to verify (1.1.1).

Applying the special case $N = 1$ of (2.4.10) to $p_{00}^{(n)}$, we have

$$p_{00}^{(n)}(s+t) - p_{00}^{(n)}(s)\, p_{00}^{(n)}(t) = \int_0^s \int_0^t p_{00}^{(n)}(s-u)\, h_n(u+v)\, p_{00}^{(n)}(t-v)\, \mathrm{d}u\, \mathrm{d}v.$$

The left-hand side is equal to

$$\sum_{\alpha=1}^{\infty} p_{0\alpha}^{(n)}(s)\, p_{\alpha 0}^{(n)}(t) = \sum_{\alpha=1}^{\infty} (p_{00}^{(n)} * g_{0\alpha}^{(n)})\,(s)\,(f_{\alpha 0}^{(n)} * p_{00}^{(n)})\,(t),$$

so that, for almost all (s, t) $\qquad h_n(s+t) = \sum_{\alpha=1}^{\infty} g_{0\alpha}^{(n)}(s) f_{\alpha 0}^{(n)}(t).$

Hence
$$\begin{aligned}\sum_{k\in I} p_{0k}(s)\, p_{k0}(t) &= p(s)\,p(t) + \sum_{\alpha, n=1}^{\infty} (p * g_{0\alpha}^{(n)})\,(s)\,(f_{\alpha 0}^{(n)} * p)\,(t) \\ &= p(s)\,p(t) + \sum_{n=1}^{\infty} \int_0^s \int_0^t p(s-u) \left\{ \sum_{\alpha=1}^{\infty} g_{0\alpha}^{(n)}(u) f_{\alpha 0}^{(n)}(v) \right\} p(t-v)\, \mathrm{d}u\, \mathrm{d}v \\ &= p(s)\,p(t) + \int_0^s \int_0^t p(s-u) \sum_{n=1}^{\infty} h_n(u+v)\, p(t-v)\, \mathrm{d}u\, \mathrm{d}v \\ &= p(s)\,p(t) + \int_0^s \int_0^t p(s-u)\, h(u+v)\, p(t-v)\, \mathrm{d}u\, \mathrm{d}v \\ &= p(s+t) = p_{00}(s+t).\end{aligned}$$

This proves the case $i = j = 0$ of (1.1.1); the remaining cases $i = 0, j = (\beta, n)$; $i = (\alpha, n), j = 0$; $i = (\alpha, n), j = (\beta, n)$ follow by very similar calculations, and the proof is complete.

Thus, in the special case $N = 1$, the converse of theorem VI is true. However, theorem VII gives an effective way of testing a function p for property M, once (3.3.1) has been solved for k:

(1) check that k is increasing and absolutely continuous,

(2) compute the maximal lower semicontinuous density h for k (cf. Kingman 1971),

(3) unless h vanishes identically, check that h is strictly positive, and at most exponentially small at infinity.

The converse of theorem VII is in fact true for general N, and therefore provides a similar recipe for testing a given $(N \times N)$ matrix-valued function for the possibility that it is a principal submatrix of some M-semigroup.

THEOREM VIII. *Suppose that h_{ij} $(i,j = 1, 2, \ldots, N)$, are lower semicontinuous functions, each of which is either identically zero or satisfies* (3.2.2), *that the $(N \times N)$ matrix-valued function $\boldsymbol{k}$ on $(0, \infty)$ is locally integrable and satisfies* (3.2.1) *with $\boldsymbol{h} = (h_{ij})$ and that $\boldsymbol{p} = (p_{ij})$ is the semi-p-matrix corresponding to $\boldsymbol{k}$ in the canonical correspondence. Then there is an M-semigroup $\boldsymbol{P}$ with index set $\{1, 2, 3, \ldots\}$ such that*

$$(\boldsymbol{P}(t))_{ij} = p_{ij}(t) \quad (i, j = 1, 2, \ldots, N). \tag{3.3.8}$$

Proof. Write $I_0 = \{1, 2, \ldots, N\}$ and for each $i, j \in I_0$ construct a countably infinite set I_{ij} such that the different I_{ij} are disjoint subsets of $\{N+1, N+2, \ldots\}$ and

$$I = I_0 \cup \bigcup_{i,j\in I_0} I_{ij} = \{1, 2, 3, \ldots\}.$$

We attempt to extend the semi-p-matrix $p = (p_{ij};\, i, j \in I_0)$ to an M-semigroup $P = (p_{ij};\, i, j \in I)$ by setting

$$p_{i\alpha} = p_{ik} * g_\alpha \quad (\alpha \in I_{kj}),$$

$$p_{\alpha j} = f_\alpha * p_{kj} \quad (\alpha \in I_{ik}),$$

$$p_{\alpha\beta} = \pi_{\alpha\beta}\delta_{jk}\delta_{il} + f_\alpha * p_{ij} * g_\beta \quad (\alpha \in I_{ki}, \beta \in I_{jl}),$$

for $i, j, k, l \in I_0$. We assert that P satisfies (1.1.1) so long as, for $s, t > 0$, $i, j \in I_0$,

$$h_{ij}(s+t) = \sum_{\alpha \in I_{ij}} g_\alpha(s) f_\alpha(t), \tag{3.3.9}$$

$$f_\alpha(s+t) = \sum_{\beta \in I_{ij}} \pi_{\alpha\beta}(s) f_\beta(t) \quad (\alpha \in I_{ij}), \tag{3.3.10}$$

$$g_\alpha(s+t) = \sum_{\beta \in I_{ij}} g_\beta(s)\, \pi_{\beta\alpha}(t) \quad (\alpha \in I_{ij}), \tag{3.3.11}$$

and

$$\pi_{\alpha\beta}(s+t) = \sum_{\gamma \in I_{ij}} \pi_{\alpha\gamma}(s)\, \pi_{\gamma\beta}(t) \quad (\alpha, \beta \in I_{ij}). \tag{3.3.12}$$

The computations needed to justify this assertion fall again into four groups, of which one will be exhibited; for $\alpha \in I_{ik}$

$$\begin{aligned}
\sum_{b \in I} p_{\alpha b}(s)\, p_{bj}(t) &= \sum_{l \in I_0} p_{\alpha l}(s)\, p_{lj}(t) + \sum_{l, m \in I_0} \sum_{\beta \in I_{lm}} p_{\alpha\beta}(s)\, p_{\beta j}(t) \\
&= \sum_{l=1}^{N} (f_\alpha * p_{kl})(s)\, p_{lj}(t) + \sum_{\beta \in I_{ik}} \pi_{\alpha\beta}(s)\, (f_\beta * p_{kj})(t) \\
&\quad + \sum_{l, m=1}^{N} \sum_{\beta \in I_{lm}} (f_\alpha * p_{kl} * g_\beta)(s)\, (f_\beta * p_{mj})(t) \\
&= \sum_{l=1}^{N} \int_0^s f_\alpha(s-u)\, p_{kl}(u)\, du\, p_{lj}(t) + \sum_{\beta \in I_{ik}} \pi_{\alpha\beta}(s) \int_0^t f_\beta(v)\, p_{kj}(t-v)\, dv \\
&\quad + \sum_{l, m=1}^{N} \int_0^s \int_0^u \int_0^t f_\alpha(s-u)\, p_{kl}(u-w) \left(\sum_{\beta \in I_{lm}} g_\beta(w) f_\beta(v) \right) p_{mj}(t-v)\, du\, dw\, dv \\
&= \int_0^s f_\alpha(s-u) \left(\sum_{l=1}^{N} p_{kl}(u)\, p_{lj}(t) \right) du + \int_0^t f_\alpha(s+v)\, p_{kj}(t-v)\, dv \\
&\quad + \int_0^s f_\alpha(s-u) \left(\int_0^u \int_0^t \sum_{l, m=1}^{N} p_{kl}(u-w)\, h_{lm}(w+v)\, p_{mj}(t-v)\, dw\, dv \right) du \\
&= \int_0^s f_\alpha(s-u) \left(\boldsymbol{p}(u)\, \boldsymbol{p}(t) + \int_0^u \int_0^t \boldsymbol{p}(u-w)\, \boldsymbol{h}(w+v)\, \boldsymbol{p}(t-v)\, dw\, dv \right)_{kj} du \\
&\quad + \int_{-t}^{0} f_\alpha(s-u)\, p_{kj}(u+t)\, du \\
&= \int_{-t}^{s} f_\alpha(s-u)\, p_{kj}(u+t)\, du = p_{\alpha j}(s+t).
\end{aligned}$$

This proves that (1.1.1) holds for $i \notin I_0$, $j \in I_0$; the other three cases are similar. To prove the theorem, it therefore suffices to choose the f, g, π so that (3.3.9), (3.3.10), (3.3.11), (3.3.12) and

$$\lim_{t \to 0} \pi_{\alpha\beta}(t) = \delta_{\alpha\beta} \quad (\alpha, \beta \in I_{ij}) \tag{3.3.13}$$

are satisfied, since then P will be an M-semigroup satisfying (3.3.8).

Now confine attention to a particular pair $i, j \in I_0$, and write $I_1 = I_{ij}$, $h = h_{ij}$. Because of what has been assumed about h, theorem VII shows that there is an M-semigroup $\boldsymbol{P}^1$ with index set $\{\partial\} \cup I_1$ such that $p = p^1_{\partial\partial}$ satisfies (3.3.1) and (3.3.2). Define

$$f_\alpha = f^1_{\alpha\partial}, \quad g_\alpha = g^1_{\partial\alpha}, \quad \pi_{\alpha\beta} = {}_\partial p^1_{\alpha\beta}$$

for $\alpha, \beta \in I_{ij}$. Then (3.3.12) and (3.3.13) follow from theorem V, and

$$p^1_{\alpha\partial} = f_\alpha * p, \quad p^1_{\partial\alpha} = p * g_\alpha.$$

Hence
$$p(s+t) - p(s)\,p(t) = \sum_{\alpha \in I_1} (p * g_\alpha)(s)\,(f_\alpha * p)(t),$$

whence (3.3.9) follows by now familiar calculations.

It remains only to prove (3.3.10) and (3.3.11). Let J be any finite subset of I_1, and $\boldsymbol{k}$ the matrix corresponding to the semi-p-matrix

$$\boldsymbol{p} = (p^1_{\alpha\beta};\, \alpha, \beta \in J \cup \{\partial\}).$$

Then, for $\alpha, \beta \in J$,
$$p^1_{\alpha\beta} = \delta_{\alpha\beta} + \sum_{\gamma \in J} (p^1_{\alpha\gamma} * k_{\gamma\beta}) + (p^1_{\alpha\partial} * k_{\alpha\beta}),$$

so that, recalling the definition of the taboo function $\pi_{\alpha\beta}$,

$$\begin{aligned}\sum_{\eta \in J} p^1_{\alpha\eta} * \pi_{\eta\beta} &= (1 * \pi_{\alpha\beta}) + \sum_{\gamma, \eta \in J} (p^1_{\alpha\gamma} * k_{\gamma\eta} * \pi_{\eta\beta}) + p^1_{\alpha\partial} * \sum_{\eta \in J} (k_{\partial\eta} * \pi_{\eta\beta}) \\ &= (1 * \pi_{\alpha\beta}) + \sum_{\gamma \in J} p^1_{\alpha\gamma} * (\pi_{\gamma\beta} - \delta_{\gamma\beta}) + (f_\alpha * p * \sum_{\eta \in J} k_{\partial\eta} * \pi_{\eta\beta}).\end{aligned}$$

Hence, using (2.4.7), we have

$$1 * p^1_{\alpha\beta} = (1 * \pi_{\alpha\beta}) + (f_\alpha * p * 1 * g_\beta),$$

so that
$$p^1_{\alpha\beta} = \pi_{\alpha\beta} + f_\alpha * p * g_\beta. \tag{3.3.14}$$

Since this is true for all $\alpha, \beta \in J$, and J is any finite subset of I_1, it is true for all α, $\beta \in I_1$. Therefore

$$\begin{aligned}(f_\alpha * p)(s+t) &= p^1_{\alpha\partial}(s+t) \\ &= \sum_{\beta \in I_1} p^1_{\alpha\beta}(s)\,p^1_{\beta\partial}(t) + p^1_{\alpha\partial}(s)\,p^1_{\partial\partial}(t) \\ &= \sum_{\beta \in I_1} (\pi_{\alpha\beta} + f_\alpha * p^1_{\partial\beta})(s)\,p^1_{\beta\partial}(t) + (f_\alpha * p)(s)\,p(t), \\ &= \sum_{\beta \in I_1} \pi_{\alpha\beta}(s)\,p^1_{\beta\partial}(t) + \int_0^s f_\alpha(u)\,\{\sum_{\beta \in I_1} p^1_{\partial\beta}(s-u)\,p^1_{\beta\partial}(t) + p(s-u)\,p(t)\}\,\mathrm{d}u \\ &= \sum_{\beta \in I_1} \pi_{\alpha\beta}(s)\,p^1_{\beta\partial}(t) + \int_0^s f_\alpha(u)\,p(s+t-u)\,\mathrm{d}u.\end{aligned}$$

Hence
$$\int_s^{s+t} f_\alpha(u)\,p(s+t-u)\,\mathrm{d}u = \sum_{\beta \in I_1} \pi_{\alpha\beta}(s)\,p^1_{\beta\partial}(t),$$

so that
$$\int_0^t f_\alpha(s+t-v)\,p(v)\,\mathrm{d}v = \sum_{\beta \in I_1} \pi_{\alpha\beta}(s) \int_0^t f_\beta(t-v)\,p(v)\,\mathrm{d}v.$$

Inverting the convolution transform, we have

$$f_\alpha(s+t) = \sum_{\beta \in I_1} \pi_{\alpha\beta}(s) f_\beta(t),$$

which is (3.3.10). The dual equation (3.3.11) is proved in the same way, and the theorem is therefore proved.

3.4. *The localization principle*

In order to establish the localization principle described in § 1.3, it is first necessary to extend theorem VIII to give the conditions under which $\boldsymbol{P}$ may be chosen to be, not just an M-semigroup, but a Markov semigroup. The next theorem justifies an assertion to this end made without proof in (Kingman 1972).

THEOREM IX. *Under the conditions of theorem VIII, the M-semigroup* $\boldsymbol{P}$ *may be chosen to be a Markov semigroup if and only if*

$$\boldsymbol{k}(t)\,\boldsymbol{1} \leqslant \boldsymbol{0} \tag{3.4.1}$$

for all $t > 0$.

Proof. It is clear from the proof of theorem II that (3.4.1) is the necessary and sufficient condition for the semi-p-matrix $\boldsymbol{p}$ to be a p-matrix. Hence (3.4.1) is certainly necessary for $\boldsymbol{P}$ to be a Markov semigroup.

Conversely, suppose that (3.4.1) holds, so that $\boldsymbol{p}$ is a p-matrix, and in particular

$$\sum_{j=1}^{N} p_{ij}(t) \leqslant 1$$

for all $t > 0$ and all i in $1 \leqslant i \leqslant N$. For any $h > 0$, define

$$\begin{aligned} x_i(h) &= 1 \quad (1 \leqslant i \leqslant N) \\ &= \sum_{n=1}^{\infty} \Sigma\, p_{ik_1}(h)\, p_{k_1 k_2}(h) \ldots p_{k_{n-1} k_n}(h) \quad (i \geqslant N+1), \end{aligned}$$

where $\boldsymbol{P} = (p_{ij})$ is the M-semigroup constructed in the proof of theorem VIII, and the second summation extends over all $k_1, k_2, \ldots, k_{n-1} \geqslant N+1$ and $k_n \leqslant N$. Then (1.1.1) shows that

$$x_i(2h) \leqslant x_i(h)$$

and that

$$x_i(h) \geqslant \sum_{j=1}^{\infty} p_{ij}(h)\, x_j(h).$$

Iterating this inequality, using (1.1.1) again, we have

$$x_i(h) \geqslant \sum_{j=1}^{\infty} p_{ij}(hn)\, x_j(h).$$

Hence

$$x_i = \lim_{r\to\infty} x_i(2^{-r})$$

exists, and setting $n = [t/h]$ and using the continuity of p_{ij}, we have

$$x_i \geqslant \sum_{j=1}^{\infty} p_{ij}(t)\, x_j$$

for all $t > 0$. This shows that $x_j < \infty$ except perhaps for values of j with

$$j \geqslant N+1, \quad p_{ij}(t) = 0 \quad \text{for all} \quad i \leqslant N.$$

Thus, if

$$Z_\infty = \{j; x_j = \infty\},$$

then

$$p_{ij} = 0 \quad (i \notin Z_\infty, j \in Z_\infty).$$

Similarly, if

$$Z_0 = \{k; x_j = 0\},$$

then

$$p_{ij} = 0 (i \in Z_0, j \notin Z_0).$$

Now define

$$p^{+}_{ij}(t) = p_{ij}(t)\, x_j/x_i \quad (i,j \notin Z_0 \cup Z_\infty)$$
$$= 1 \quad (i = j \in Z_0 \cup Z_\infty)$$
$$= 0 \quad \text{otherwise.}$$

Then it is trivial to check that $\boldsymbol{P}^{+}$ is a Markov semigroup, and since $x_i = 1$ for $i \leqslant N$ it satisfies (3.3.8).

THEOREM X. *Let* $\boldsymbol{P}$ *be an M-semigroup with index set* I, T *a positive number, and* S *a finite subset of* $I \times I$. *Then there exists a constant* α *and a Markov semigroup* $\tilde{\boldsymbol{P}}$ *on* I, *such that*

$$p_{ij}(t) = \tilde{p}_{ij}(t)\, \mathrm{e}^{\alpha t} \tag{3.4.2}$$

for $(i,j) \in S$, $t \leqslant T$.

Proof. Since $S \subseteq I \times I$ is finite, there is a finite $J \subseteq I$ with $S \subseteq J \times J$, and it clearly suffices to prove (3.4.2) for $i, j \in J$. Since the case when I is finite is trivial, there is no loss of generality in supposing that

$$I = \{1, 2, 3, \ldots\}, \quad J = \{1, 2, \ldots, N\}$$

for some finite N. Let

$$\boldsymbol{p} = \boldsymbol{p}^J = (p_{ij};\, i, j \in J),$$

and let $\boldsymbol{k}$ correspond to $\boldsymbol{p}$ in the canonical correspondence. Then, as in the proof of theorem II, the semi-p-matrix

$$\boldsymbol{p}_\alpha(t) = \boldsymbol{p}(t)\, \mathrm{e}^{-\alpha t}$$

corresponds to

$$\boldsymbol{k}_\alpha(t) = -\alpha \boldsymbol{I} + \boldsymbol{k}(t)\, \mathrm{e}^{-\alpha t} + \alpha \int_0^t \boldsymbol{k}(s)\, \mathrm{e}^{-\alpha s}\, \mathrm{d}s.$$

Choose α so large that all the elements of

$$\boldsymbol{k}_\alpha(T)\, \mathbf{1} = -\alpha \mathbf{1} + \{\boldsymbol{k}(T)\, \mathbf{1}\}\, \mathrm{e}^{-\alpha T} + \int_0^T \{\boldsymbol{k}(s)\, \mathbf{1}\}\, \mathrm{e}^{-\alpha s}\, \mathrm{d}s$$

are strictly negative.

Since $\boldsymbol{p}_\alpha$ is a submatrix of the M-semigroup $\boldsymbol{P}(t)\, \mathrm{e}^{-\alpha t}$, $\boldsymbol{k}_\alpha$ satisfies the conditions described in theorem VI, and it is therefore clear that we can define $\tilde{\boldsymbol{k}}$ in such a way that

(*a*) $\tilde{\boldsymbol{k}}(t) = \boldsymbol{k}_\alpha(t)$ for $t \leqslant T$,

(*b*) $\tilde{\boldsymbol{k}}(t)\, \mathbf{1} \leqslant \mathbf{0}$ for $t > 0$,

(*c*) $\tilde{\boldsymbol{k}}$ satisfies the conditions of theorem VI.

Theorem III and (*a*) now imply (3.4.2), and theorems VIII and IX imply the existence of a Markov semigroup $\tilde{\boldsymbol{P}}$ with

$$\tilde{\boldsymbol{p}} = \tilde{\boldsymbol{P}}^J.$$

Hence the theorem is proved.

The most important case is that in which S has a single element (i,j), where $i = j$ or $i \neq j$.

Corollary. If p_{ij} *is any element, diagonal or non-diagonal, of an M-semigroup, then on any finite interval it coincides with the product of an exponential function and an element of a Markov semigroup.*

Thus any local property of Markov transition functions (such as the continuous differentiability asserted by Ornstein's theorem) extends automatically to the elements of M-semigroups, wild as well as tame. Moreover, the same is true of finite collections of such elements.

It is important to note that the converse of theorem X is false, even in the simplest case of a single diagonal element, and even in the Markov case. For example, let p be the p-function corresponding to

$$k(t) = -\int_t^\infty \mathrm{e}^{-x^2}\, \mathrm{d}x.$$

Then p does not have property M because (3.2.2) is violated, but for any finite T there is a p-function with property M, for instance that corresponding to

$$k_T(t) = -\int_t^\infty e^{-x^2}\,dx \quad (t \leqslant T)$$

$$= -\int_T^\infty e^{-x^2}\,dx\, e^{-(t-T)} \quad (t > T),$$

which agrees with p on $[0, T]$.

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